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Local Lagrange interpolation by quintic C^1 splines on type-6 tetrahedral partitions

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ABSTRACT

We develop a Lagrange interpolation method for quintic C^1 splines on cube partitions with 24 tetrahedra in each cube. The construction of the interpolation points is based on a new priority principle by decomposing the tetrahedral partition into special classes of octahedra such that no tetrahedron has to be refined. It follows that the interpolation method is local and stable, and has optimal approximation order six and linear complexity. The interpolating splines are uniquely determined by data values, but no derivatives are needed.

1. Introduction

We develop a Lagrange interpolation method for quintic C^1 splines on type-6 tetrahedral partitions. The type-6 tetrahedral partition is derived from a uniform cube partition by intersecting each cube with six planes, resulting in 24 tetrahedra per cube [1]. This partition can be considered as a collection of octahedra, with each octahedron consisting of eight tetrahedra. The method is based on a new priority principle, in which the partition is decomposed into classes of octahedra with an increasing number of common vertices, edges, and faces. In contrast to the known methods [2–6], no tetrahedron has to be refined, and no additional smoothness conditions are required. The interpolating spline is constructed step by step on each octahedron, starting with the octahedra in the lowest class. We show that the method is local and stable in the sense that the value of each B-coefficient of a given octahedron depends only on data values located in the neighborhood of the octahedron and is absolutely bounded by these data values (25). In our main result, we show that the interpolation method has approximation order six (26), which is optimal for quintic splines.

The paper is organized as follows. In Section 2, we recall some basic facts and notation about trivariate splines on tetrahedral partitions, and Bézier–Bernstein techniques. In Section 3, we investigate the space of quintic C^1 supersplines on type-6 tetrahedral partitions, and determine the dimension of this spline space. We decompose the partition into classes of octahedra in Section 4. Based on this decomposition, we construct a Lagrange interpolation set for the spline space in Section 5. It is shown that the method is local and stable. In Section 6, we establish error bounds for the interpolating splines, and show that the method has optimal approximation order six.

2. Preliminaries

We first recall some basic facts about trivariate splines on tetrahedral partitions and Bézier–Bernstein techniques.

Given a polygonal domain $\Omega \subset \mathbb{R}^3$, a *tetrahedral partition* Δ of Ω is a set of non-degenerate tetrahedra $\{T_1, \ldots, T_n\}$ where $\bigcup_{i=1}^n T_i = \Omega$ and the intersection of any two tetrahedra $T, \widetilde{T} \in \Delta, T \neq \widetilde{T}$ is either empty, a common vertex, a common

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edge, or a common face. The *space of trivariate splines of degree d and smoothness r* on Δ is defined as

$$\mathcal{S}_d^r(\Delta) := \left\{s \in C^r(\Omega);\ s_{|T} \in \mathcal{P}_d \text{ for all tetrahedra } T \in \Delta\right\},$$

where $r \le d$ are positive integers and

$$\mathcal{P}_d := \operatorname{span}\left\{x^i y^j z^k;\ 0 \le i+j+k \le d,\ i,j,k \in \mathbb{N}\right\}$$

is the $\binom{d+3}{3}$-dimensional space of trivariate polynomials of total degree d.

We use the well-known Bézier–Bernstein representation of splines. Let $T \in \Delta$, $T := \Delta(v_0, v_1, v_2, v_3)$ be some tetrahedron with vertices $v_0, \ldots, v_3$; then the set of *domain points* on T is defined as

$$\mathcal{D}_T := \left\{\xi_{ijkl} := (iv_0 + jv_1 + kv_2 + lv_3)/d;\ i+j+k+l = d\right\}.$$

The *ball* of radius r around the vertex v_0 with respect to T is the subset of domain points defined by

$$D_r^T(v_0) := \left\{\xi_{i,j,k,l} \in \mathcal{D}_T;\ i \ge d-r\right\},$$

with similar definitions for the other vertices. We further define

$$D_r(v) := \bigcup_{T \in \Delta;\ v \in T} D_r^T(v).$$

For a tetrahedral partition Δ, we define

$$\mathcal{D}_\Delta := \bigcup_{T \in \Delta} \mathcal{D}_T.$$

The polynomial pieces of s are given in the *Bézier–Bernstein representation*,

$$s_{|T} :\equiv \sum_{i+j+k+l=d} b_{ijkl} B_{ijkl},$$

where the coefficients b_{ijkl} are the *B-coefficients* of $s_{|T}$ and

$$B_{ijkl} :\equiv \frac{d!}{i!j!k!l!} \varphi_0^i \varphi_0^j \varphi_0^k \varphi_0^l$$

is the *Bernstein polynomial* of degree d associated with T. The *barycentric coordinates* $\varphi_\nu \in \mathcal{P}_1$, $\nu = 0, \ldots, 3$, are the unique linear polynomials satisfying $\varphi_\nu(v_\mu) = \delta_{\nu\mu}$, $\nu, \mu = 0, \ldots, 3$, where $\delta_{\nu\mu}$ denotes Kronecker's delta. Since the Bernstein polynomials form a basis for $\mathcal{P}_d$, $s_{|T}$ is uniquely determined by its B-coefficients b_{ijkl}, $i+j+k+l = d$. We associate the B-coefficient b_{ijkl} and the Bernstein polynomial B_{ijkl} with the respective domain point ξ_{ijkl}. For a domain point $\xi \in \mathcal{D}_T$, we denote the B-coefficient of $s_{|T}$ associated with ξ by b_ξ, and the Bernstein polynomial associated with ξ by B_ξ. Let $T = \Delta(v_0, v_1, v_2, v_3)$ and $\widetilde{T} = \Delta(v_0, v_1, v_2, \widetilde{v}_3)$ be two neighboring tetrahedra, and let $s \in \mathcal{S}_d^{-1}(T \cup \widetilde{T})$. For $i+j+k+l = d$, let b_{ijkl}^T and $b_{ijkl}^{\widetilde{T}}$ be the B-coefficients of $s_{|T}$ and $s_{|\widetilde{T}}$, respectively, and let B_{ijkl}^T be the Bernstein polynomials associated with T. Then $s \in \mathcal{S}_d^r(T \cup \widetilde{T})$ if and only if the B-coefficients satisfy

$$b_{ijk\rho}^{\widetilde{T}} = \sum_{i_0+j_0+k_0+l_0=\rho} b_{i+i_0,j+j_0k+k_0,l_0}^T B_{i_0j_0k_0l_0}^T, \quad i+j+k = d-\rho, \tag{1}$$

for all $\rho = 0, \ldots, r$. Note that, if the B-coefficients associated with the ball $D_r^T(v)$ for some vertex v of T are known, then the remaining B-coefficients associated with $D_r(v)$ are determined by this equation.

Let Δ be a tetrahedral partition and let v be a vertex of Δ. Then the *star of* v is defined as the set of tetrahedra which share the vertex v,

$$\operatorname{star}(v) := \{T \in \Delta;\ v \in T\}.$$

For a tetrahedron $T \in \Delta$, we further define $\operatorname{star}^0(T) := T$ and

$$\operatorname{star}^\ell(T) := \left\{\widetilde{T} \in \Delta;\ \widetilde{T} \cap \operatorname{star}^{\ell-1}(T) \ne \emptyset\right\}.$$

We also make use of the concept of a minimal determining set for a spline space. Let $\mathcal{S}(\Delta)$ be a spline space defined on a tetrahedral partition Δ, and let $\mathcal{M} \subseteq \mathcal{D}_\Delta$ be a subset of the domain points of Δ. $\mathcal{M}$ is called a *determining set* for $\mathcal{S}(\Delta)$ if $b_\xi = 0$ for all $\xi \in \mathcal{M}$ implies that $s \equiv 0$. $\mathcal{M}$ is called a *minimal determining set* (MDS) if no subset of $\mathcal{M}$ with fewer elements is a determining set for $\mathcal{S}(\Delta)$. It is well known that, if $\mathcal{M}$ is an MDS for $\mathcal{S}(\Delta)$, then $\dim(\mathcal{S}(\Delta)) = \#\mathcal{M}$ [7, p. 485, Theorem 17.8].