Journal of Computational and Applied Mathematics 236 (2011) 543–556

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction

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ARTICLE INFO

Keywords:
Non-stationary subdivision
Symbol
Exponential polynomial
Exponential generation
Exponential reproduction

ABSTRACT

We present an accurate investigation of the algebraic conditions that the symbols of a non-singular, univariate, binary, non-stationary subdivision scheme should fulfill in order to reproduce spaces of exponential polynomials. A subdivision scheme is said to possess the property of reproducing exponential polynomials if, for any initial data uniformly sampled from some exponential polynomial function, the scheme yields the same function in the limit. The importance of this property is due to the fact that several curves obtained by combinations of exponential polynomials (such as conic sections, spirals or special trigonometric and hyperbolic functions) are considered of interest in geometric modeling. Since the space of exponential polynomials trivially includes standard polynomials, this work extends the theory on polynomial reproduction to the non-stationary context. A significant application of the derived algebraic conditions on the subdivision symbols is the construction of new non-stationary subdivision schemes with specific reproduction properties.

1. Introduction

Non-stationary subdivision schemes are efficient iterative algorithms to construct special classes of curves ranging from polynomials and trigonometric curves to conic sections or spirals. The aim of this paper is to establish the algebraic conditions that fully identify the reproduction properties of a given non-singular, univariate, binary, non-stationary subdivision scheme.

Following the notation in [1], we denote by $\{\mathbf{a}^{(k)},\ k \geq 0\}$ the sequence of finite sets of real coefficients identifying, for each k, the *k-level mask* of a non-stationary subdivision scheme, and we define by

$$a^{(k)}(z) = \sum_{j\in\mathbb{Z}} a_j^{(k)} z^j, \quad k \geq 0,\ z \in \mathbb{C} \setminus \{0\}, \tag{1}$$

the Laurent polynomial whose coefficients are exactly the entries of $\mathbf{a}^{(k)}$. The polynomial in (1) is commonly known as the *k-level symbol* of the non-stationary subdivision scheme associated with $\mathbf{a}^{(k)}$.

Hereinafter, we will denote through $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$ the sequence of linear subdivision operators based on the masks $\{\mathbf{a}^{(k)},\ k \geq 0\}$, identifying the non-stationary subdivision scheme

$$\mathbf{f}^{(k+1)} := S_{\mathbf{a}^{(k)}} \mathbf{f}^{(k)}, \quad (S_{\mathbf{a}^{(k)}} \mathbf{f}^{(k)})_i := \sum_{j\in\mathbb{Z}} a_{i-2j}^{(k)} f_j^{(k)},\ k \geq 0, \tag{2}$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.03.031

starting from any initial "data" sequence $\mathbf{f}^{(0)} \equiv \mathbf{f} := \{f_i \in \mathbb{R},\ i \in \mathbb{Z}\}$. Note that, whenever the recursive relation in (2) relies on the same mask at each level of refinement, namely $\mathbf{a}^{(k)} = \mathbf{a}$ for all $k \geq 0$, then the subdivision scheme is said to be stationary.

In the non-stationary case, one could even start the subdivision process with a mask at level $m \geq 0$, so getting a family of subdivision schemes based on the masks $\{\mathbf{a}^{(m+k)},\ k \geq 0\},\ m \geq 0$.

Attaching the data $f_i^{(k)}$ generated at the kth step to the parameter values $t_i^{(k)}$ with

$$t_i^{(k)} < t_{i+1}^{(k)}, \quad \text{and} \quad t_{i+1}^{(k)} - t_i^{(k)} = 2^{-k}, \quad k \geq 0,$$

we see that the subdivision process generates denser and denser sequences of data so that a notion of convergence can be established by taking into account the piecewise linear function $F^{(k)}$ that interpolates the data, namely

$$F^{(k)}(t_i^{(k)}) = f_i^{(k)}, \qquad F^{(k)}|_{[t_i^{(k)}, t_{i+1}^{(k)}]} \in \Pi_1, \quad i \in \mathbb{Z},\ k \geq 0,$$

where Π_1 is the space of linear polynomials. If the sequence $\{F^{(k)},\ k \geq 0\}$ converges, then we denote its limit by

$$g_{\mathbf{f}}^m := \lim_{k\to\infty} S_{\mathbf{a}^{(m+k)}} S_{\mathbf{a}^{(m+k-1)}} \cdots S_{\mathbf{a}^{(m)}} \mathbf{f}^{(0)} = \lim_{k\to\infty} F^{(k)}, \quad m \geq 0,$$

and say that $g_{\mathbf{f}}^m$ is the *limit function* of the non-stationary subdivision scheme (2) when starting with the mask at level m, for the data $\mathbf{f}$ (while we simply use $g_{\mathbf{f}}$ in the stationary case).

If for all $m \geq 0$ the non-stationary subdivision scheme is convergent, and $g_{\mathbf{f}}^m \equiv 0$ if and only if $\mathbf{f} \equiv 0$, then the subdivision scheme is termed *non-singular*. In the forthcoming discussion we restrict ourselves to non-singular schemes only.

Since most of the properties of a subdivision scheme (e.g. its convergence, its smoothness or its support size) do not depend on the choice of the parameter values $t_i^{(k)}$, these are usually set as

$$t_i^{(k)} := \frac{i}{2^k}, \quad i \in \mathbb{Z},\ k \geq 0. \tag{3}$$

Following [2,3] we refer to the choice in (3) as to the "standard" parametrization. As it will be better clarified later, with respect to the subdivision capability of reproducing specific classes of functions, the standard parametrization is not always the optimal one. Indeed, for example the choice

$$t_i^{(k)} := \frac{i+p}{2^k}, \quad i \in \mathbb{Z},\ p \in \mathbb{R},\ k \geq 0, \tag{4}$$

with p suitably set, turns out to be a better selection. The parameter setting in (4) corresponds to the general formula $t_i^{(k)} = t_0^{(0)} - p + \frac{i+p}{2^k}$ proposed in [2], where we select the free parameter $t_0^{(0)} = p$, such that the data $f_{2^k}^{(k)}$ is attached to the integers in the limit. Note also that, when $p = 0$ the parametrization in (4) is called "primal", while in the case $p = -\frac{1}{2}$ "dual". For a complete discussion concerning the choice of parametrization in the analysis of polynomial reproduction for stationary subdivision schemes, we refer the reader to the papers [2,3]. In passing, we recall the following definition since it lays the foundations of our results in the non-stationary situation.

Definition 1. A convergent, stationary subdivision scheme $S_{\mathbf{a}}$ is *generating* polynomials up to degree d_G if for any polynomial π of degree $d \leq d_G$ there exists some initial data $\mathbf{q}^{(0)}$ such that $g_{\mathbf{q}^{(0)}} = S_{\mathbf{a}}^{\infty}\mathbf{q}^{(0)} = \pi$. Moreover, $\mathbf{q}^{(0)}$ is sampled from a polynomial of the same degree and with the same leading coefficient. Additionally, a convergent subdivision scheme $S_{\mathbf{a}}$ is *reproducing* polynomials up to degree d_R if for any polynomial π of degree $d \leq d_R$ and for the initial data $\boldsymbol{\pi}^{(0)} = \{\pi(t_i^{(0)}),\ i \in \mathbb{Z}\}$ it holds $g_{\boldsymbol{\pi}^{(0)}} = S_{\mathbf{a}}^{\infty}\boldsymbol{\pi}^{(0)} = \pi$.

We also recall that a convergent subdivision scheme that reproduces polynomials of degree d_R has approximation order $d_R + 1$ (see [4] for the proof of this result). This fact motivates the systematic investigation of polynomial reproduction, as emphasized in [2,3].

In this context, the first purpose of this paper is to deal with the extension of the concepts of polynomial generation and polynomial reproduction to the non-stationary setting, leading to the concepts of exponential polynomial generation and exponential polynomial reproduction. In a few words, a subdivision scheme is said to possess the property of reproducing exponential polynomials if, for any initial data uniformly sampled from some exponential polynomial function, the scheme yields the same function in the limit. This is a very desirable property of a non-stationary subdivision scheme for its possible use in geometric modeling [5]. The second purpose of this paper is to show that, as in the stationary case discussed in [2], the choice of the correct parametrization is crucial in the non-stationary setting as well. Our analysis brings to algebraic conditions, involving a parametrization of the form (4), that the symbols of a non-singular, binary, non-stationary subdivision scheme should fulfill in order to reproduce specific spaces of exponential polynomials.

To illustrate the application of the simple but very general algebraic conditions derived, we construct novel subdivision symbols defining new non-stationary subdivision schemes with specific reproduction properties.