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# High-order finite element methods for time-fractional partial differential equations[☆]

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ABSTRACT

The aim of this paper is to develop high-order methods for solving time-fractional partial differential equations. The proposed high-order method is based on high-order finite element method for space and finite difference method for time. Optimal convergence rate $O((\Delta t)^{2-\alpha}+N^{-r})$ is proved for the $(r-1)$th-order finite element method $(r\geq 2)$.



## 1. Introduction

Fractional partial differential equations (PDEs) have wide applications in the real world (see e.g., in [1,2] and [3]) and thus the solutions of the equations become increasingly popular (see e.g., in [1,4–6] and the listed references). In this paper, we study one type of time-fractional PDEs, which can be obtained from the standard parabolic PDEs by replacing the first-order time derivative with a fractional derivative of order $\alpha$, $0<\alpha<1$. More precisely, we consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),\quad (x,t)\in[0,1]\times[0,T] \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0)=u_0(x),\quad x\in I=[0,1], \tag{2}$$

$$u(0,t)=u(1,t)=0,\quad t\in(0,T], \tag{3}$$

where $0<\alpha<1$, $f$ and $u_0$ are given smooth functions and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}.$$

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The analytical solutions of the time-fractional PDEs are studied using Green's functions or Fourier–Laplace transforms (see e.g., in [1,7–9]). However, the references for the numerical methods are very limited. Most existing methods are lower-order methods, for example, Liu et al. [10] study the first-order finite difference methods; Scherer et al. [11] develop Grünwald–Letnikov's approach (a variant of finite difference method), analyze the stability and discuss the convergence rates.

As pointed out in paper [12], it is necessary to develop high-order methods due to the fractional term. High-order methods—spectral methods are studied by Lin and Xu [12]. Lin and Xu [12] (in Theorem 4.2, 4.3) show that the methods for $\alpha$-order time-fractional partial differential equations with $0 < \alpha < 1$ have convergence rate $O(\Delta t^{2-\alpha} + N^{-m}/(\Delta t)^{\alpha})$, where $m$ measures the regularity of the solution in space. Obviously the convergence rates in their paper are not optimal due to the impairment of the factor $(\Delta t)^{-\alpha}$. In this paper, we use high-order finite element methods to solve the same equation and prove an optimal convergence rate. Since the finite element methods use piecewise polynomial bases not like the spectral methods using global polynomial bases, the finite element methods are much easier to implement.

In the rest of the paper, we assume that the solution $u$ is sufficiently smooth. We use the following norms: $\|v\| = \|v\|_{L^2(I)}$ and $\|v\|_r = \|v\|_{H^r(I)}$. $C$ denotes a generic positive constant that is independent of mesh but depends on the smoothness of $u$.

## 2. High-order finite element methods

Let $\tau = T/L$ be the time meshsize, $t_n = n\tau, n = 0, 1, \dots, L$ be mesh points and $t_{n-1/2} = \frac{t_{n-1}+t_n}{2}, n = 1, 2, \dots, L$, be mid mesh points, where $L$ is a positive integer. The time-fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial^\alpha t}$ at $t_n$ is estimated by

$$\begin{aligned}
\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial u(x,s)}{\partial s} \frac{\mathrm{d}s}{(t_n - s)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{\partial}{\partial t} u(x, t_{k-1/2}) \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}s}{(t_n - s)^\alpha} + \gamma_n^{(1)}(x) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{\partial}{\partial t} u(x, t_{n-k-1/2}) \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}s}{s^\alpha} + \gamma_n^{(1)}(x) \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{\partial}{\partial t} u(x, t_{n-k-1/2}) + \gamma_n^{(1)}(x),
\end{aligned} \tag{4}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ and

$$\gamma_n^{(1)}(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial u(x,s)}{\partial s} \frac{\mathrm{d}s}{(t_n - s)^\alpha} - \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{\partial u(x, t_{k-1/2})}{\partial t} \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}s}{(t_n - s)^\alpha}.$$

Let $h = 1/N$ and use the uniform space mesh with mesh points

$$x_i = ih, \quad i = 0, 1, \dots, N.$$

Denote $S_h$ the set of piecewise polynomials of degree at most $r-1$ on mesh $\{x_i\}$. Define Ritz projection $R_h$ from $H_0^1(I)$ into $S_h$ by the orthogonal relation

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_h,\ v \in H_0^1(I).$$

Define

$$\gamma_n^{(2)}(x) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \left( \frac{\partial}{\partial t} u(x, t_{n-k-1/2}) - \frac{R_h u(x, t_{n-k}) - R_h u(x, t_{n-k-1})}{\tau} \right).$$

Then combining with (4), we have

$$\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{R_h u(x, t_{n-k}) - R_h u(x, t_{n-k-1})}{\tau} + \gamma_n(x), \tag{5}$$

with $\gamma_n(x) = \gamma_n^{(1)}(x) + \gamma_n^{(2)}(x)$.

The weak form of (1)–(3) is given by

$$\left( \frac{\partial^\alpha}{\partial t^\alpha} u, \phi \right) + a(u, \phi) = (f, \phi), \quad \forall \phi \in H_0^1(I), \tag{6}$$