Journal of Computational and Applied Mathematics 235 (2011) 3304–3314

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Comparison of certain value-at-risk estimation methods for the two-parameter Weibull loss distribution

Omer L. Gebizlioglu [a,*], Birdal Şenoğlu [a], Yeliz Mert Kantar [b]

[a] Department of Statistics, Faculty of Science, Ankara University, 06100 Tandoğan, Ankara, Turkey

[b] Department of Statistics, Faculty of Science, Anadolu University, Eskişehir, Turkey

ARTICLE INFO

*Article history:*
Received 13 April 2010
Received in revised form 13 August 2010

*Keywords:*
Value-at-risk
Quantiles
Weibull distribution
Monte Carlo simulation
Deficiency

ABSTRACT

The Weibull distribution is one of the most important distributions that is utilized as a probability model for loss amounts in connection with actuarial and financial risk management problems. This paper considers the Weibull distribution and its quantiles in the context of estimation of a risk measure called Value-at-Risk (VaR). VaR is simply the maximum loss in a specified period with a pre-assigned probability level. We attempt to present certain estimation methods for VaR as a quantile of a distribution and compare these methods with respect to their deficiency (*Def*) values. Along this line, the results of some Monte Carlo simulations, that we have conducted for detailed investigations on the efficiency of the estimators as compared to MLE, are provided.



## 1. Introduction

The Weibull distribution belongs to a class of probability distributions that are known as heavy-tailed distributions; it is even a super-exponential distribution [1, p. 23–78]. With this feature of it, the Weibull distribution is a natural choice for the probability distribution of losses with potentially high outcomes (heavy tail) in finance and insurance [2,3], mostly in the context of the ruin risk modeling. Benckert and Lung [4], Beirlant and Teugels [5] and Mikosch [6] give stimulating ideas and examples that emphasize the heavy-tailed distributional modeling regarding the actuarial science and insurance applications.

By definition, a risk measure is a mapping from a set of random variables, that stand for risks, to the real numbers. The motivation behind the use of risk measures in finance and actuarial science is to determine some critical quantities like required capital, risk reserves and premiums.

Value-at-Risk (VaR) is one of the oldest risk measures that has been intensively used in finance and insurance business. It is basically defined as the maximum expected loss for a given probability. VaR, as a risk measure, is seen very practical to use in many real life risk management practices but it is also criticized for some of its inadequacies in the measurement of risk and risk oriented decision making [7]. Even so, it is such an essential measure that many comprehensive risk measures like Tail Value-at-Risk (TVaR), Conditional Tail Expectation (CTE), Conditional VaR (CVaR) and Expected Shortfall (ES) of a random variable (r.v.) can be expressed as a function of the VaRs of that r.v. [8–10].

Given a loss random variable $X$, say an aggregate claim amount for an insurance portfolio, with probability distribution function $F_X(x)$, VaR at level $\alpha$ $(0 \le \alpha \le 1)$, denoted by $\mathrm{VaR}_\alpha(X)$, is actually an $\alpha$-quantile risk measure for $X$. Formally, the $\alpha$-quantile risk measure for $X$ is defined as $Q_\alpha[X] = \inf\{x : F_X(x) \ge \alpha\}$ which is a left-continuous and non-decreasing

* Corresponding author. Tel.: +90 312 2126720 1258; fax: +90 312 2233202.
*E-mail address:* gebizli@ankara.edu.tr (O.L. Gebizlioglu).

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doi:10.1016/j.cam.2011.01.044

function of $\alpha$. For all real valued $X$ and $\alpha$ values, there is an equivalence relation between the quantiles and distribution functions; $Q_\alpha[X] \leq x \leftrightarrow \alpha \geq F_X(x)$. Then, by definition,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha), \tag{1.1}$$

where $F_X^{-1}(\alpha)$ is the inverse of function $F_X(x)$ at a chosen $\alpha$ value.

The subjects of VaRs and quantiles of r.v.'s have been tackled by many authors. Among the recent ones of them, Embrechts et al. [11], Denuit et al. [9] and Gebizlioglu and Yagcı [12] present some results about these quantities for the dependent risks.

Due to the overwhelming significance of quantiles both in statistical theory and risk theory and in their applications, as well, the estimation of $Q_\alpha[X]$ and thus $\mathrm{VaR}_\alpha(X)$ has been always a crucial matter of study. Thereupon, there have been various attempts to obtain efficient estimators of $\mathrm{VaR}_\alpha(X)$ which can be through the parametric approaches at portfolio or position level or through the nonparametric approaches [13, p. 1740–1747]. The nonparametric estimator and the maximum likelihood estimator (MLE) are the two most commonly used estimators of $\mathrm{VaR}_\alpha(X)$ [14]. The nonparametric method of estimating the $\alpha$-quantile of $F$ is known to be the method of sample quantiles. It is well known and widely used in statistical inference and applications [15]. The $\alpha$-quantile of $F$ in the sample quantile context is defined as $X_{n,[n\alpha]+1}$ where $X_{n,[n\alpha]+1}$ is the $([n\alpha]+1)$th order statistic of the sample and [.] is the greatest integer value function. On the other hand; for a given value of $\alpha$, the MLE of $\mathrm{VaR}_\alpha(X)$ is calculated by inserting the MLE estimators of the parameters into the inverse function $F^{-1}$.

In this paper, we consider different estimation methods for $\mathrm{VaR}_\alpha(X)$ and study how they behave for different shape parameters and for different sample sizes that are drawn from a Weibull distribution as a loss distribution for a loss amount random variable. We compare the performances of nine parametric estimators including MLE with respect to their *Def* values through Monte Carlo simulation. *Def* is a mean square error (MSE) based measure of the joint efficiency of estimators of a set of parameters, say $(\theta, \beta)$, of a probability distribution. It is calculated as the sum of MSEs, $Def(\hat{\theta}, \hat{\beta}) = \mathrm{MSE}(\hat{\theta}) + \mathrm{MSE}(\hat{\beta})$, for the estimates $(\hat{\theta}, \hat{\beta})$, which are obtained by a chosen method of estimation. As the methods of estimation for the parameters at hand, we compute and compare moment estimator (ME), generalized spacing estimator (GSE), modified maximum likelihood estimator I (MMLE-I), modified maximum likelihood estimator II (MMLE-II), Tiku's modified maximum likelihood estimator (TMMLE), least squares estimator (LSE), weighted least squares estimator (WLSE) and percentile estimator (PE).

Recently, Kantar and Şenoğlu [16] have estimated the parameters of the Weibull distribution using the estimators mentioned above with known shape parameter and compared them with respect to their bias, MSE and *Def* values.

A new approach has been implemented in our work: a two-parameter Weibull loss distribution with a known location parameter is considered and some different estimators of its scale and the shape parameters are compared with respect to the *Def* values. For the first time in the research area, the best method that gives the highest efficiency for estimating $\mathrm{VaR}_\alpha(X)$ is shown here for several sample sizes and shape parameter values.

The paper is arranged as follows: Section 2 introduces the Weibull distribution. A concise description of the estimators mentioned above is given in Section 3. Section 4 presents the results of the simulation study. Section 5 presents a real life example with data from the insurance sector. The conclusions are given in Section 6.

## 2. The Weibull distribution

Cumulative distribution function (CDF) of the two-parameter Weibull distribution is given by

$$F(x; \theta, \beta) = 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\}, \quad x \geq 0\, \theta > 0,\ \beta > 0, \tag{2.1}$$

where $\theta$ is the scale parameter and $\beta$ is the shape parameter. The corresponding pdf is expressed by

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{x}{\theta}\right)^\beta}, \quad x \geq 0,\ \theta > 0,\ \beta > 0. \tag{2.2}$$

The Weibull distribution is reversed J shaped if $\beta < 1$, and bell shaped if $\beta > 1$. When $\beta = 1$, it reduces to the density function of the well-known exponential distribution.

Among the distributions with asymmetric features, the Weibull distribution is used in many applications in the areas of risk management, insurance, finance life testing, reliability engineering and biology, as well, because of its flexible properties mentioned above.

Under the two-parameter Weibull loss distribution, $\mathrm{VaR}_\alpha(X)$ is estimated by

$$\widehat{\mathrm{VaR}}_\alpha(X) = \{-\ln(1-\alpha)\}^{1/\hat{\beta}}\, \hat{\theta} \tag{2.3}$$

where $\alpha$ is the confidence level.

It is clear that the estimation and the efficiency of estimators for VaR rely on the estimation of the parameters of the Weibull distribution considered here.

Many methods have been proposed to estimate the unknown parameters of the Weibull distribution. MLE is the most widely used method among the others because it has the asymptotic efficiency property under a parametric model. However,