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On accurate product integration rules for linear fractional differential equations

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ABSTRACT

This paper addresses the numerical solution of linear fractional differential equations with a forcing term. Competitive and highly accurate Product Integration rules are derived by starting from an equivalent formulation in terms of a Volterra integral equation with a generalized Mittag-Leffler function in the kernel. The error analysis is reported and aspects related to the computational complexity are treated. Numerical tests confirming the theoretical findings are presented.

1. Introduction

The current spread of fractional calculus in new applications is remarkable; fractional differential equations (FDEs) are indeed commonly used to describe and simulate models coming from several areas, such as probability theory, biology, economics and physics.

The strength of derivatives of non-integer order is their capability to describe real situations more adequately than integer order derivatives, especially when the problem has memory or hereditary properties. We just refer to [1] for a deep description of the subject and for a complete list of references.

In this paper we consider the linear FDE

$$\begin{cases} D^{\alpha}_{t_0} y(t) + \lambda y(t) = f(t) \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$ is the fractional order, $\lambda \in \mathbb{R}$, $y(t) : [t_0, T] \to \mathbb{R}$ and the forcing term $f(t)$ is assumed sufficiently smooth. For simplicity throughout this paper we will assume $0 < \alpha < 1$, although most of the results that we will provide can be easily generalized to $\alpha > 0$. Here $D^{\alpha}_{t_0}$ denotes the fractional derivative operator, with respect to the origin t_0, according to the Caputo definition [2,1]

$$D^{\alpha}_{t_0} y(t) \equiv \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y'(s)}{(t-s)^{\alpha}} \mathrm{d}s,$$

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where $\Gamma(\cdot)$ is the Euler gamma function. The approach of Caputo is the most interesting from a practical point of view since it allows us to couple the FDE with an initial condition of Cauchy type as in (1). Moreover this operator is strictly connected with the Riemann–Liouville operator

$$^{R}D^{\alpha}_{t_0}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\left(t-u\right)^{-\alpha}y(s)\mathrm{d}s$$

by means of the relationship $D^{\alpha}_{t_0}y(t) = {}^{R}D^{\alpha}_{t_0}y(t) - \frac{1}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}y(t_0)$ (e.g., see [2,1]).

Recently, the numerical solution of FDEs has been largely investigated and several methods have been proposed. Product Integration (PI) rules, originally due to the work of Young [3], are a class of convolution quadratures which are particularly interesting for the problem under investigation for the easy way in which weights can be evaluated. However, as showed by several authors [4,5] and discussed in Section 2, these rules do not have good convergence properties, especially when $\alpha < 1$. In this case the value $\alpha + 1$ can be considered as an order barrier for PI rules, thus preventing the development of highly accurate methods.

The main aim of this paper is to investigate an alternative formulation of linear FDEs (1) in order to overcome this order barrier. In Section 2 PI rules are described and their main features are reviewed. In Section 3 we introduce a reformulation of the problem in terms of a Volterra Integral Equation (VIE) with a modified kernel; its main properties are recalled. In Section 4 PI rules for the reformulated problem are introduced and accuracy is investigated. Section 5 is devoted to the numerical evaluation of the kernel in the modified VIE reformulation of the FDE. Finally, in Section 6 we present some numerical tests to validate the theoretical results and we make comparisons of the methods discussed in the paper.

2. Product integration rules

It is a well-known result (e.g. [2,1]) that problem (1) can be rewritten as a weakly singular VIE of second type

$$y(t) = \phi(t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,y(s))\mathrm{d}s \tag{2}$$

where $g(t,y(t)) = -\lambda y(t) + f(t)$ and $\phi(t) \equiv y_0$ when $0 < \alpha < 1$.

The reformulation in terms of VIEs is a useful tool thanks to which several classes of numerical methods have been developed for solving FDEs.

Product Integration (PI) is a widely used technique for numerically solving VIEs and we briefly recall its idea: given an equispaced grid $t_n = t_0 + nh$ on $[t_0, T]$, with step-size $h > 0$, the function $g(s, y(s))$ in (2) is replaced by a suitable piecewise interpolant polynomial and the resulting integrals are evaluated exactly. Sometimes the term *k-step* PI rule is used to highlight that polynomials of degree k replace the function g; even though this notation is useful (and we will continue to use it in this paper) we remark that methods of this type are not k-step in the classical sense since all the previous evaluated values are used during the computation and not only the last k values.

One of the most studied PI rules in the context of FDEs (e.g., see [6,4,7,8]) is the 1-step (or trapezoidal) rule given by

$$y_n = y_0 + h^{\alpha}a_n\, g(t_0,y_0) + h^{\alpha}\sum_{j=1}^{n-1}\alpha_{n-j}\, g(t_j,y_j) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\, g(t_n,y_n), \tag{3}$$

with $a_n = \frac{(n-1)^{\alpha+1}-n^{\alpha}(n-\alpha-1)}{\Gamma(\alpha+2)}$ and $\alpha_n = \frac{(n-1)^{\alpha+1}-2\,n^{\alpha+1}+(n+1)^{\alpha+1}}{\Gamma(\alpha+2)}$ for $n > 0$.

Cameron and McKee [9] first investigated the order of convergence of PI methods for second kind Abel integral equations and found that, under the assumption of sufficient continuity of the true solution on the whole interval of integration $[t_0, T]$, implicit k-step PI methods converge with order $p = k + 1$.

However, the required assumptions of regularity are very seldom satisfied. In [10] Lubich studied smoothness properties and showed that the true solution of (2) has an asymptotic expansion in mixed powers of $t - t_0$ and $(t - t_0)^{\alpha}$. Since the presence of a discontinuity already on the first derivatives at the origin, PI rules can fail to converge with full order. For instance, it has been showed [5] that, when applied to second kind weakly singular linear VIEs, PI rules converging with order $p \geq 2$ under smoothness assumptions on the true solution, in the most general case exhibit a slower convergence of order $1 + \alpha$. Similar conclusions were reached in [4] where a detailed analysis of errors for nonlinear FDEs under different smoothness hypothesis was presented. For this reason, PI rules based on polynomial of degree greater than 1 are not usually taken into consideration for FDEs and $p = 1 + \alpha$ can be considered as an order barrier for PI rules when applied to FDEs in the most general case.

3. Reformulating linear FDEs

The particular nature of the problem suggests an alternative way for reformulating the FDE (1).

By using the Laplace transform, the result given in [1] for the true solution of linear FDEs with the Riemann–Liouville derivative can be generalized to the Caputo approach to obtain the following variation of constant formula

$$y(t) = e_{\alpha,1}(t-t_0;\lambda)y_0 + \int_{t_0}^{t}e_{\alpha,\alpha}(t-s;\lambda)f(s)\mathrm{d}s \tag{4}$$