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The approximate solution of a class of Fredholm integral equations with a weakly singular kernel

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ABSTRACT

A method for finding the numerical solution of a weakly singular Fredholm integral equation of the second kind is presented. The Taylor series is used to remove singularity and Legendre polynomials are used as a basis. Furthermore, the Legendre function of the second kind is used to remove singularity in the Cauchy type integral equation. The integrals that appear in this method are computed in terms of gamma and beta functions and some of these integrals are computed in the Cauchy principal value sense without using numerical quadratures. Four examples are given to show the accuracy of the method.

1. Introduction

Weakly singular integral equations have many applications in problems of mathematical physics. These equations arise in the heat conduction problem posed by mixed boundary conditions [1], potential problems, the Dirichlet problem, and radiative equilibrium [2]. Furthermore, Badr in [3] has cited important applications of weakly singular integral equations in the fields of fracture mechanics, elastic contact problems, the theory of porous filtering, combined infrared radiation and molecular conduction. It is difficult to solve these equations analytically and analytical solutions in some special cases can be found in [4–7]. Hence, numerical solutions are required. Recently, numerical solutions for these equations have been developed by many authors and researchers. Lifanov in [8] introduced hypersingular integral equations with applications and a numerical solution for a class of these equations of Prandtl's type is given in [6]. Numerical solutions for the Cauchy and Abel type of weakly singular integral equations are discussed in [1–3,6–16]. The polar kernel of integral equations has been introduced in [2,16]. This singular kernel is in the form

$$k(x,y)=\frac{g(x,y)}{(x-y)^{\alpha}}+h(x,y),\quad 0<\alpha\le 1$$

where the first term of this kernel is weakly singular and g and h are bounded on the square $s=[-1,1]\times[-1,1]$ and $g(x,y)\neq 0$. With $g=1$, $h=0$, we have the special case of the above kernel

$$k(x,y)=\frac{1}{(x-y)^{\alpha}},\quad 0<\alpha\le 1.$$

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One of the weakly singular integral and integro-differential equations with this kernel has been introduced in [9,3,11]. We introduce the following singular integral equation

$$\mu(x)\phi(x) + \lambda(x) \int_{-1}^{1} \frac{\phi(y)}{(y-x)^\alpha} \, dy = f(x), \quad |x| < 1, \ 0 < \alpha \leq 1, \tag{1.1}$$

where $\mu(x) \neq 0, \lambda(x) \neq 0$ and $\mu(x), \lambda(x), f(x) \in L^2[-1, 1]$ are given functions and $\phi(x)$ is the unknown function to be determined. In [9,3,11], Eq. (1.1) has been considered but $\mu(x)$ and $\lambda(x)$ are taken to be constants. With $\alpha = 1$, we have the Cauchy type singular integral equation and in this case we suppose that the integral in Eq. (1.1) exists in the Cauchy principal value sense. Let

$$I_\alpha(x) = \int_{-1}^{1} \frac{\phi(y)}{(y-x)^\alpha} \, dy = f(x), \quad |x| < 1, \ 0 < \alpha \leq 1.$$

If $\phi(x)$ is Hölder continuous in the interval $[-1, 1]$ then $I_\alpha(x)$ exists. In other words, if there exist $A > 0, 0 < m < 1$ such that

$$\forall x, y \in [-1, 1], \quad |\phi(x) - \phi(y)| \leq A|x-y|^m,$$

then

$$I_\alpha(x) = \int_{-1}^{1} \frac{\phi(y)-\phi(x)}{(y-x)^\alpha} \, dy + \phi(x) \int_{-1}^{1} \frac{dy}{(y-x)^\alpha}, \quad |x| < 1, \ 0 < \alpha < 1.$$

When $\alpha = 1$, we have

$$I_\alpha(x) = \int_{-1}^{1} \frac{\phi(y)-\phi(x)}{y-x} \, dy + \phi(x) \int_{-1}^{1} \frac{dy}{y-x}, \quad |x| < 1,$$

where the second integral exists in the Cauchy principal value sense and

$$\text{p.v.} \int_{-1}^{1} \frac{dy}{y-x} = \ln\left(\frac{1-x}{1+x}\right), \quad |x| < 1.$$

If $\phi(y)$ is differentiable on the interval $[-1, 1]$ and $\phi'(y)$ is bounded on this interval, by using the mean value theorem $I_\alpha(x)$ will exist. Here we consider the integral equation given in relation (1.1) and assume that $\phi(y)$ has the Taylor series expansion of any order on the interval $(-1, 1)$.

2. Numerical solution

Consider the integral equation with the given conditions in relation (1.1). With the Taylor series expansion of $\phi(y)$ based on expanding about the given point x belonging to the interval $I = (-1, 1)$, we have the Taylor series approximation of $\phi(y)$ in the following form

$$\phi(y) = \phi(x) + (y-x)\phi'(x) + \frac{(y-x)^2}{2!}\phi''(x) + \cdots + \frac{(y-x)^n}{n!}\phi^{(n)}(x) + \frac{(y-x)^{n+1}}{(n+1)!}\phi^{(n+1)}\left(\zeta_{x,y}\right), \tag{2.1}$$

where $\zeta_{x,y}$ is between x and y. By substituting relation (2.1) into Eq. (1.1), we have

$$\begin{aligned} &\mu(x)\phi(x) + \lambda(x) \int_{-1}^{1} \left(\frac{1}{(y-x)^\alpha} \sum_{k=0}^{n} \frac{(y-x)^k}{k!} \phi^{(k)}(x) \right) dy + E_n(x) \\ &\quad = \mu(x)\phi(x) + \lambda(x) \sum_{k=0}^{n} \frac{\phi^{(k)}(x)}{k!} \int_{-1}^{1} (y-x)^{k-\alpha} \, dy + E_n(x) = f(x), \end{aligned} \tag{2.2}$$

where

$$\phi^{(0)}(x) = \phi(x),$$

$$E_n(x) = \frac{1}{(n+1)!} \int_{-1}^{1} (y-x)^{n+1-\alpha} \phi^{(n+1)}\left(\zeta_{x,y}\right) \, dy.$$

Alternatively, we use the truncated Taylor series of $\phi(y)$ and solve the following equation

$$\mu(x)\phi(x) + \lambda(x) \sum_{k=0}^{n} \frac{\phi^{(k)}(x)}{k!} I_{\alpha,k}(x) = f(x), \tag{2.3}$$