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Gerber–Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times

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ABSTRACT

In this paper, we consider a compound Poisson risk model perturbed by a Brownian motion. We construct the bivariate cumulative distribution function of the claim size and interclaim time by Farlie–Gumbel–Morgenstern copula. The integro-differential equations and the Laplace transforms for the Gerber–Shiu functions are obtained. We also show that the Gerber–Shiu functions satisfy some defective renewal equations. For exponential claims, some explicit expressions are obtained, and numerical examples for the ruin probabilities are also given.

1. Introduction

Consider the following compound Poisson risk model that is perturbed by a Brownian motion

$$U(t) = u + ct - S(t) + \sigma B(t), \tag{1.1}$$

where $u \geq 0$ is the initial surplus and $c > 0$ is the premium rate. The aggregate claims process $S(t) = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson process, where $\{N(t), t \geq 0\}$ is a Poisson process denoting the number of claims up to time t, and $\{X_i, i \geq 1\}$ is a sequence of strictly positive random variables representing the individual claim sizes. The interclaim times $\{V_i, i \geq 1\}$ is a sequence of exponential random variables distributed like a generic variable V with probability density function (p.d.f.) $k(t) = \lambda e^{-\lambda t}$ for $\lambda > 0$, cumulative distribution function (c.d.f.) $K(t) = 1 - e^{-\lambda t}$. Finally, $B(t)$ independent of the aggregate claims process is a standard Brownian motion starting from zero, and $\sigma > 0$ is the diffusion volatility.

The compound Poisson risk model perturbed by a diffusion process was first proposed in [1] to extend the classical risk model. Since then, it has received a lot of attention, and in such risk model, many ruin problems including the evaluation of the Gerber–Shiu discounted penalty functions have been well studied, see e.g. [2–7]. However, in all the aforementioned papers, it is assumed that the claim sizes $\{X_i\}$ are independent of the interclaim times $\{V_i\}$. Although such an assumption indeed simplifies the analysis of the ruin problems, it has been proved that the independence assumption is somewhat restrictive in modeling some real situations. As for the risk model with dependence but without diffusion, many researchers have made contributions to it, see e.g. [8–12]. Zhou and Cai [13] consider a perturbed risk model where the premium rates depend on the claim size which is an extension of the model considered in [8]. We note that Zhou and Cai's [13]'s model has the stationary and independent increments property in the Markovian sense. Recently, Zhang and Yang [14] add a diffusion to the dependent risk model of [9] and study the ruin probabilities by using a potential measure.

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In this paper, we will study a perturbed risk model with dependence, where the claim size and interclaim time has a certain bivariate c.d.f. The rest of this paper is organized as follows. In Section 2, we describe the dependence structure and introduce the ruin measures. The roots of a Lundberg-type equation are analyzed in Section 3. We show that the integro-differential equations for the Gerber–Shiu functions can be obtained in Section 4. In Section 5, the Laplace transforms for the Gerber–Shiu functions are given by using the roots of the Lundberg-type equation, and as a by-product, defective renewal equations for the Gerber–Shiu functions are also obtained. Finally, explicit expressions and numerical results are given for exponential claims in Section 6.

2. Dependence structure and ruin measures

Motivated in [12], we use the Farlie–Gumbel–Morgenstern (FGM) copula to define the dependence structure between the claim size and the interclaim time. The FGM copula is given by

$$C_{FGM}(u_1, u_2) = u_1u_2 + \theta u_1u_2(1 - u_1)(1 - u_2), \quad 0 \le u_1,\ u_2 \le 1, \tag{2.1}$$

where $-1 \le \theta \le 1$. Note that FGM copula allows both negative and positive dependence, and it also includes the independence copula ($\theta = 0$). For more on the FGM copula, we refer to [15,11,12].

Let X be a strictly positive random variable with p.d.f. f and c.d.f. F. The bivariate c.d.f. of (X, V) with marginals F and K is given by

$$F_{X,V}(x, t) = C_{FGM}(F(x), K(t)), \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

And the joint p.d.f. of (X, V) is given by

$$f_{X,V}(x, t) = c_{FGM}(F(x), K(t))f(x)k(t), \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

where $c_{FGM}(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C_{FGM}(u_1, u_2)$. Given (2.1), we have

$$F_{X,V}(x, t) = F(x)K(t) + \theta F(x)K(t)(1 - F(x))(1 - K(t)), \tag{2.2}$$

$$f_{X,V}(x, t) = \lambda e^{-\lambda t}f(x) + \theta(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t})h(x), \tag{2.3}$$

where $h(x) = (1 - 2F(x))f(x)$.

In the rest of this paper, we assume that $\{(X_i, V_i), i \ge 1\}$ form a sequence of i.i.d. random vectors distributed like (X, V) with joint c.d.f. and p.d.f. respectively given by (2.2) and (2.3). In particular, we know from (2.3) that the conditional p.d.f. of the claim size is given by

$$f_{X|V=t}(x) = f(x) + \theta(2e^{-\lambda t} - 1)h(x). \tag{2.4}$$

Also, we assume that $\theta \ne 0$, since otherwise our model reduces to the classical risk model perturbed by diffusion.

Associated with the risk model (1.1) is the ruin time, τ, which is the first passage time of $U(t)$ below zero level, i.e.

$$\tau = \inf\{t \ge 0, U(t) < 0\},$$

with $\tau = \infty$ if $U(t) \ge 0$ for all $t \ge 0$. The ultimate ruin probability is defined by

$$\psi(u) = \Pr(\tau < \infty | U(0) = u). \tag{2.5}$$

By observing the sample paths of $U(t)$, we know that ruin can be caused either by the oscillation of the Brownian motion or a downward jump. Similar to [2], we could decompose the probability of ruin as follows

$$\psi(u) = \psi_w(u) + \psi_d(u), \tag{2.6}$$

where $\psi_w(u)$ is the ruin probability when ruin is caused by a claim, and $\psi_d(u)$ is the ruin probability when ruin is due to oscillation. To guarantee that ruin is not a certain event, we assume that the following net profit condition holds

$$E[cV - X] > 0. \tag{2.7}$$

By careful calculations, we can check that (2.7) is equivalent to

$$c > \lambda \int_0^\infty xf(x)\mathrm{d}x. \tag{2.8}$$

In order to study more ruin measures, we introduce the Gerber–Shiu function defined by

$$\phi(u) = E[e^{-\delta\tau} w(U(\tau-), |U(\tau)|)I(\tau < \infty)|U(0) = u], \tag{2.9}$$

where $\delta \ge 0$ is the force of interest, $I(\cdot)$ is the indicator function, $w(x_1, x_2)$ defined on $[0, \infty) \times [0, \infty)$ is a nonnegative function of the surplus before ruin $U(\tau-)$ and the deficit at ruin $|U(\tau)|$. Similarly, $\phi(u)$ can also be decomposed according to whether ruin is caused by a claim or oscillation, i.e.

$$\phi(u) = \phi_w(u) + \phi_d(u), \tag{2.10}$$