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Qualitative analysis of a stochastic ratio-dependent predator–prey system[☆]

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ABSTRACT

A stochastic ratio-dependent predator–prey model is investigated in this paper. By the comparison theorem of stochastic equations and Itô's formula, we obtain the global existence of a positive unique solution of the ratio-dependent model. Besides, a condition for species to be extinct is given and a persistent condition is established. We also conclude that both the prey population and the ratio-dependent function are stable in time average. In the end, numerical simulations are carried out to confirm our findings.

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. The classical models are mostly variations of the Lotka (a physical chemist)–Volterra (a mathematician) model which is the product of chemistry, physics and mathematics. Specially, it is the product of the (chemistry) principle of mass action, (physics) laws of conservation, and (mathematics) elemental differential equations. Obviously, the biological content is the missing link. Inevitably, the traditional models have been challenged by several biologists (see, for example, [2–4]) based on the fact that functional and numerical responses over typical ecological timescales ought to depend on the densities of both predators and prey (most likely and simply on their ratio), especially when predators have to search for food (and therefore have to share or compete for food). Such a functional response is called a ratio-dependent response function and these hypotheses have been strongly supported by numerous fields and laboratory experiments and observations [2,3,5–7]. Based on the Michaelis–Menten or Holling type II function, Arditi and Ginzburg [2] first proposed a ratio-dependent function of the form

$$P\left(\frac{x}{y}\right) = \frac{cx/y}{m + x/y} = \frac{cx}{my + x},$$

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and the following ratio-dependent predator–prey model:

$$\begin{cases} \dot{x}(t) = x(t)\left(a - bx(t) - \dfrac{cy(t)}{my(t)+x(t)}\right), \\ \dot{y}(t) = y(t)\left(-d + \dfrac{fx(t)}{my(t)+x(t)}\right). \end{cases} \tag{1.1}$$

Here, $x(t)$ and $y(t)$ represent population densities of the prey and the predator at time t, respectively; parameters a, b, c, d, f, m are positive constants in which a/b is the carrying capacity of the prey, a, c, m, f, d stand for the prey intrinsic growth rate, capturing rate, half capturing saturation constant, conversion rate and the predator death rate, respectively. In resent years, several authors have studied the ratio-dependent predator–prey model (1.1) and its extension and have observed rich dynamics, for example, see [8–15].

On the other hand, population systems are often affected by environmental noise, and hence stochastic differential equation models play a significant role in various branches of applied sciences including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterpart [16,17]. In reality, due to continuous fluctuations in the environment (e.g. variation in intensity of sunlight, temperature, water level, etc.), parameters involved in models are not absolute constants, but they always fluctuate around some average value. As a result the population density never attains a fixed value with the advancement of time but rather exhibits continuous oscillation around some average values. Based upon these factors, stochastic population models have received more and more attention [18–20]. To the best of our knowledge a small amount of work has been done with the stochastic ratio-dependent prey–predator models. In [21], they assumed that stochastic perturbations of the state variables were $\sigma_1(x - x^*)\dot{B}_1(t)$ and $\sigma_2(y - y^*)\dot{B}_2(t)$, where (x^*, y^*) is the positive equilibrium point of system (1.1), $\dot{B}_1(t)$ and $\dot{B}_2(t)$ are white noises, and they showed that the corresponding stochastic model is asymptotically mean square stable. While in [22], considering that fluctuations in the environment would manifest themselves mainly as fluctuations in the intrinsic growth rate of the prey population and in the death rate of the predator population [23], they supposed parameters a and d were perturbed with

$$a \to a + \alpha\dot{B}_1(t), \qquad d \to d + \beta\dot{B}_2(t),$$

where $B_1(t)$ and $B_2(t)$ are mutually independent Brownian motions, α and β represent the intensities of the white noises. Then the stochastic ratio-dependent predator–prey model took the following form:

$$\begin{cases} dx(t) = x(t)\left(a - bx(t) - \dfrac{cy(t)}{my(t)+x(t)}\right)dt + \alpha x(t)dB_1(t), \\ dy(t) = y(t)\left(-d + \dfrac{fx(t)}{my(t)+x(t)}\right)dt - \beta y(t)dB_2(t). \end{cases} \tag{1.2}$$

By Laplace transform methods for stochastic differential equation model, they [22] calculated population fluctuation intensity (variance) for prey and predator species. But for population dynamics, we are more interested in the persistence and extinction of the system. As mentioned above, ratio-dependent predator–prey systems without stochastic perturbation have been well studied and much richer dynamics found. They [13] especially showed system (1.1) has equilibria $(0, 0)$, $(\frac{a}{b}, 0)$ and a unique positive equilibrium $E^* = (x^*, y^*) = (\frac{cd-f(c-ma)}{bfm}, \frac{f-d}{dm}x^*)$. System (1.1) is permanent if

$$f > d \quad \text{and} \quad ma > c, \tag{1.3}$$

while if

$$cm^{-1} > a + d, \tag{1.4}$$

then system (1.1) is not persistent, and

$$\lim_{t\to+\infty}(x(t), y(t)) = (0, 0).$$

Motivated by this, in this paper we mainly attempt to consider system (1.2), which inherits a similar property of the deterministic system, and a property which is not.

This paper is organized as follows. In Section 2, by Itô's formula and the comparison theorem of stochastic equations, we show that system (1.2) has a unique positive solution $(x(t), y(t))$ a.s. with any initial value $x(0) = x_0 > 0, y(0) = y_0 > 0$. Besides, we also find that both the prey population and the predator population of system (1.2) are bounded in mean. In Section 3, we establish if

$$a - \frac{c}{m} - \frac{\alpha^2}{2} > 0 \quad \text{and} \quad f - d - \frac{\beta^2}{2} > 0, \tag{1.5}$$

then both the prey population $x(t)$ and the ratio-dependent equation $P(\frac{x}{y})$ are stable in time average. That is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = \frac{c\left(d - \frac{\beta^2}{2}\right) - f\left(c - m\left(a - \frac{\alpha^2}{2}\right)\right)}{bfm}$$