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Numerical implementation of the EDEM for modified Helmholtz BVPs on annular domains

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ABSTRACT

In a recent paper by the current authors a new methodology called the Extended-Domain-Eigenfunction-Method (EDEM) was proposed for solving elliptic boundary value problems on annular-like domains. In this paper we present and investigate one possible numerical algorithm to implement the EDEM. This algorithm is used to solve modified Helmholtz BVPs on annular-like domains. Two examples of annular-like domains are studied. The results and performance are compared with those of the well-known boundary element method (BEM). The high accuracy of the EDEM solutions and the superior efficiency of the EDEM over the BEM, make EDEM an excellent alternate candidate to use in the animation industry, where speed is a predominant requirement, and by the scientific community where accuracy is the paramount objective.

1. Introduction

In a recent paper [1] the authors proposed a semi-analytic method for solving elliptic boundary value problems (EBVPs) on annular-like domains with boundaries involving complex geometry, the Extended-Domain-Eigenfunction method or EDEM. The method is based on the concept of considering the domain Ω of the original problem to be part of a much larger domain that possesses greater symmetry. In this paper, we investigate one possible numerical implementation of EDEM.

In the original EDEM paper, the method was also posed as an alternative to more traditional methods such as the Finite Element Method (FEM) [2,3], the Boundary Element Method (BEM) [3–11], the Finite Difference Method (FDM) [12] and the Boundary Point Method (BPM) [13–17]. These types of numerical methods are required to solve EBVPs when an analytic or semi-analytic solution cannot be obtained, due to the complexity of the boundary. EDEM provides an alternative which is much easier to implement, less numerically intense and more accurate, and due to its semi-analytic nature it also provides greater insight into the actual problem.

The methodology of EDEM [1] involves formulating a related problem on an appropriate larger or extended domain. Using the inherent symmetry of the new domain, an eigenfunction solution is generated for the related problem on the extended domain using standard techniques such as separation of variables. The problem can then be solved using a

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variety of numerical techniques (e.g. a collocation method). Finally, this solution is restricted to the original domain to obtain the solution of the original problem. In the original paper [1], we provide a formal statement of EDEM and detailed discussion about the criteria and boundary conditions that allow for the method to be used for the case of the Laplace operator.

Recently, Shankar [18,19] independently proposed a similar method to EDEM, based on the method of eigenfunction expansions (MEE). Much like EDEM, a BVP on a complicated geometric domain is embedded in a larger domain endowed with a regular boundary and a complete set of eigenfunctions, allowing for an eigenfunction solution to be formulated. The solution of this embedded problem could then be used to effect the solution of the original problem. The method was applied to solve a series of simply and multiply connected domains for a variety of problems. However, as Shankar [18] admitted in his paper some important theoretical issues were not addressed and also the method assumed that the original domain was convex and that a nice extension for the solution existed. Some of these theoretical issues have now been addressed in [1]. Shankar also left open questions regarding the comparison of the method with other solution techniques, in particular the Boundary Element Method. Issues about computational efficiency and ease of implementation were only briefly discussed. This latter issue is what we address in the present publication. As was pointed out in [1], EDEM is not limited to the particular numerical implementation advocated in this paper, i.e., a collocation based approach. Shankar has demonstrated this fact by using a least squares approach. This difference in numerical schemes demonstrates a very important fact that there is no unique numerical approach associated with the theoretical method.

When numerical implementations are considered, they can be seen to overlap with another semi-analytic approach known as the Trefftz method [20–22]. The term Trefftz method is used to describe a series of diverse and different methods, all derived from Trefftz's original idea [20]. These methods also utilise eigenfunctions of the differential operator to construct a finite sum approximation to the EBVP. Therefore, overlap between our implementation and earlier ones occurs when the eigenfunction solution is truncated in the EDEM algorithm. In the literature, the Trefftz methods are referred to as collocation based Trefftz methods. An in-depth discussion of these methods can be found in [23]. It is important to point out that the EDEM methodology as outlined in [1] provides a theoretical basis even for these Trefftz methods and other related numerical schemes for problems on annular-like domains. In light of this overlap and to avoid any ambiguity in the literature we shall refer to the implementation followed in this paper as the EDEM–Trefftz algorithm.

A significant volume of literature and research has been published on the Trefftz method. In particular over the past decade there has been a resurgence in interest in the method. The potential for a mesh free algorithm with fast computational times makes it an area of interest both within and outside of the academic community. However, traditionally the Trefftz method is seen to be limited by the scope of domains it can handle. Typically, it has been restricted to singly connected domains of simple geometry, whereas as FEM and the BEM have been accepted as being more applicable to generic domains. Huang and Shaw [24] improved on the standard collocation Trefftz method by applying the embedding integral approach to the eigenfunction expansion method. By doing so and choosing partitions based on the problem's domain, they were able to solve Laplace and Helmholtz problems on a variety of different domains. Also, some recent works have expanded on this idea by looking into multi-pole Trefftz methods for solving Helmholtz and modified Helmholtz problems on multiply connected circular domains [25,26].

In this paper we will investigate and implement an EDEM–Trefftz numerical algorithm. We consider and solve EBVPs for the case of the modified Helmholtz equation

$$\Delta u - \kappa^2 u = 0, \quad x \in \Omega. \tag{1}$$

This elliptic operator is chosen for two reasons. Firstly, the modified Helmholtz operator is seldom studied in this context, while most of the literature dedicated to semi-analytic numerical solutions of EBVPs has concentrated on other more well-known classical PDE operators such as the Laplace [21,27–32], Poisson [21,33,34] and standard Helmholtz [21,35–37,26,25] equation, even in the case of the Trefftz method. This study will therefore add to the literature on this modified Helmholtz operator [22,38,10,25] as well as provide insight into the recently proposed EDEM and much older Trefftz method. This will complement the current body of work looking at applications of the BEM to the modified Helmholtz BVP [8–11]. Second, the extendability criteria (including the characterisation of the operator which maps the original boundary conditions for the original problem to the new boundary conditions) have been fully explored for the case of the Laplace operator. Since the modified Helmholtz equation is closely related to the Laplace equation (it degenerates to the Laplace equation as $\kappa \to 0$), the extendability criteria and the arguments to determine them are very similar (although we do not include them here). As there are no known closed form solutions for the domains we considered, we compare the EDEM–Trefftz algorithm with the BEM since these two techniques share some similarities.

The organisation of this paper is as follows. In Section 2 we review and present an overview of the EDEM methodology for the case of a modified Helmholtz EBVP on an annular domain. We present a general example for a 2D annular domain, formulate the related problem on the extended domain and present its general solution. In Section 3, we outline the EDEM–Trefftz algorithm. Finally in Section 4, we present numerical results for two examples of annular domains involving elliptic and square inner boundaries. These results are compared for both numerical accuracy and simulation efficiency with those of the BEM using linear elements. It is shown that the EDEM–Trefftz algorithm not only produces good results which appear to be more accurate than those of the BEM, it also demonstrates a significant advance on the BEM runtimes. We also show that similar findings are found to be consistent for boundary conditions that violate the EDEM domain extension conditions [1].