Journal of Computational and Applied Mathematics 235 (2011) 1482–1489

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# The distributional products on spheres and Pizetti's formula

C.K. Li [a,*], M.A. Aguirre [b]

[a] Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada R7A 6A9

[b] Núcleo Consolidado de Matemática Pura y Aplicada, Facultad de Ciencias Exactas, UNCentro, Pinto 399, 7000 Tandil, Argentina

ARTICLE INFO

*Article history:*
Received 13 March 2009

*MSC:*
primary 46F10

*Keywords:*
Distribution
Product
Asymptotic expansion
Differential form
Fourier transform and Pizetti's formula

ABSTRACT

The distribution $\delta^{(k)}(r-a)$ concentrated on the sphere $O_a$ with $r-a=0$ is defined as

$$(\delta^{(k)}(r-a),\ \phi) = \frac{(-1)^k}{a^{n-1}}\int_{O_a}\frac{\partial^k}{\partial r^k}(\phi r^{n-1})\mathrm{d}\sigma.$$

Taking the Fourier transform of the distribution and the integral representation of the Bessel function, we obtain asymptotic expansions of $\delta^{(k)}(r-a)$ for $k=0,1,2,\ldots$ in terms of $\triangle^j\,\delta(x_1,\ldots,x_n)$, in order to show the well-known Pizetti formula by a new method. Furthermore, we derive an asymptotic product of $\phi(x_1,\ldots,x_n)\,\delta^{(k)}(r-a)$, where $\phi$ is an infinitely differentiable function, based on the formula of $\triangle^m(\phi\psi)$, and hence we are able to characterize the distributions focused on spheres, which can be written as the sums of multiplet layers in the Gel'fand sense.



## 1. Introduction

The sequential method [1] and the complex analysis approach [2], including non-standard analysis [3], have been the main tools used for dealing with products, powers and convolutions of distributions, such as $\delta^2$, which is needed when calculating the transition rates of certain particle interactions in physics [4]. Fisher (see [5–10], for example) has actively used the Jones $\delta$-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ where $\rho(x)$ is a fixed infinitely differentiable function on $R$ with the following properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = 0$ for $|x| \geq 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1}\rho(x)\mathrm{d}x = 1$,

and the concept of the neutrix limit of van der Corput [11] to deduce numerous products, powers, convolutions, and compositions of distributions on $R$ since 1969. The technique of neglecting appropriately defined infinite quantities and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Fisher's method of computation can be regarded as a particular application of the neutrix calculus. This is a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in the context of distributions by Fisher in connection with the problem of distributional multiplication, convolution and composition. To extend such an approach from the one-dimensional case to the $n$-dimensional case, Li et al. [12–15] constructed several workable $\delta$-sequences on $R^n$ for non-commutative neutrix products such as $r^{-k}\cdot\nabla\delta$ as well as $r^{-k}\cdot\triangle^l\,\delta$, where $\triangle$ denotes the Laplacian. Aguirre [16] used the Laurent series expansion of $r^\lambda$ and derived a more general product $r^{-k}\cdot\nabla(\triangle^l\,\delta)$ by calculating the residue of $r^\lambda$. His approach represents another interesting example of using complex analysis to obtain products of distributions on $R^n$.

* Corresponding author. Tel.: +1 204 571 8549; fax: +1 204 728 7346.
*E-mail addresses:* lic@brandonu.ca (C.K. Li), maguirre@exa.unicen.edu.ar (M.A. Aguirre).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.08.032

The problem of defining products of distributions on a manifold (the unit sphere is a particular example) has been a serious challenge since Gel'fand [17] introduced generalized functions of special types, such as $P_+^{\lambda}$ and $\delta^{(k)}(P)$, where

$$(\delta^{(k)}(P), \phi) = (-1)^k \int_{P=0} \omega_k(\phi).$$

Li [18] studied the products $f(P)\delta^{(k)}(P)$ and $f(P,Q)\delta(PQ)$ on regular manifolds along the differential form line. Furthermore, he used the delta sequence and the convolution given for $P = 0$ to derive an invariant theorem, that powerfully converts the products of distributions on manifolds into well defined products of a single variable. Several examples, including the products of $P_+^r(x)$ and $\delta^{(k)}(P(x))$, are presented using the invariant theorem. Aguirre [19] employed the Taylor expansion of the distribution $\delta^{(k-1)}(m^2+P)$ and gave a meaning to the product $\delta^{(k-1)}(m^2+P)\cdot\delta^{(l-1)}(m^2+P)$. In [20], Li obtained a regular product $f(r)\cdot\delta^{(k)}(r-1)$ on $\Omega$ $(=O_1)$, as well as computing several new products related to $\delta(x)$ on even-dimension spaces by a complex analysis method. Recently, Li [21] applied Pizetti's formula and a recursive structure of $\triangle^j(X^l\phi(x))$ to compute the product $X^l\delta(r-1)$. As outlined in the abstract, the goal of this work is to attempt to obtain a generalized product of $\phi(x_1,\ldots,x_n)\delta^{(k)}(r-a)$, where $\phi$ is an infinitely differentiable function, based on the following formula:

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi.$$

This enables us to expand every functional $f$ of the form

$$(f,\phi) = \int_{r=a} \sum_j a_j(x) D^j \phi(x) \mathrm{d}\sigma$$

as an infinite expansion in the distributional sense.

## 2. Pizetti's formula

We let $\mathcal{D}(R^n)$ be the Schwartz space of the testing functions with bounded support in $R^n$ and let $r^2 = \sum_{i=1}^n x_i^2$. The distribution $\delta(r-a)$ concentrated on the sphere $O_a$ with $r - a = 0$ is defined as

$$(\delta(r-a), \phi) = \int_{O_a} \phi \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the Euclidean element on the sphere $r - a = 0$.

We define $S_\phi(r)$ as the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius $r$ by

$$S_\phi(r) = \frac{1}{\Omega_n}\int_\Omega \phi(r\sigma)\mathrm{d}\sigma$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the area of the unit sphere $\Omega$ $(=O_1)$. We can write out asymptotic expression for $S_\phi(r)$, namely

$$\begin{aligned} S_\phi(r) &\sim \phi(0) + \frac{1}{2!}S_\phi''(0)r^2 + \cdots + \frac{1}{(2k)!}S_\phi^{(2k)}(0)r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k k! n(n+2)\cdots(n+2k-2)} \quad (\triangle \text{ is the Laplacian}) \end{aligned}$$

which is the well-known Pizetti formula and it plays an important role in the work of Li et al. [12,22–24]. Recently, it served as a foundation for building the gravity formula for the algebra (see [25]).

To the authors' knowledge, Pizetti's formula has not been proved as a convergent series for $\phi \in \mathcal{D}(R^n)$ since it appeared in [26]. Now we are going to show that it does converge by using the Fourier transform and the following formula which can be found in [27]:

$$J_\nu(x) = \frac{1}{2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi \mathrm{e}^{\mathrm{i}x\cos\theta} x^\nu \sin^{2\nu}\theta \mathrm{d}\theta. \tag{1}$$

The Fourier transform of $\delta(r-a)$ is defined as

$$F(\delta(r-a)) = (\delta(r-a), \mathrm{e}^{\mathrm{i}(x,\sigma)}) = \int_{O_a} \mathrm{e}^{\mathrm{i}(x,\sigma)}\mathrm{d}x.$$

In spherical coordinates ($r = |x| = a$, $\rho = |\sigma|$ and $\theta$ is the angle between the $x$ and $\sigma$ vectors) this becomes

$$\begin{aligned} F(\delta(r-a)) &= \int \mathrm{e}^{\mathrm{i}a\rho\cos\theta} a^{n-1}\sin^{n-2}\theta \mathrm{d}\theta\mathrm{d}\omega \\ &= a^{n-1}\Omega_{n-1}\int_0^\pi \mathrm{e}^{\mathrm{i}a\rho\cos\theta}\sin^{n-2}\theta\mathrm{d}\theta, \end{aligned}$$

where $\mathrm{d}\omega$ is the element of area on the unit sphere in the $(n-1)$-dimensional subspace orthogonal to $\rho$.