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# Local convergence of Newton's method under majorant condition

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ABSTRACT

A local convergence analysis of Newton's method for solving nonlinear equations, under a majorant condition, is presented in this paper. Without assuming convexity of the derivative of the majorant function, which relaxes the Lipschitz condition on the operator under consideration, convergence, the biggest range for uniqueness of the solution, the optimal convergence radius and results on the convergence rate are established. Besides, two special cases of the general theory are presented as applications.



## 1. Introduction

Newton's method and its variant are powerful tools for solving nonlinear equations in real or complex Banach space. In the past few years, a couple of papers have dealt with the issue of local and semi-local convergence analysis of Newton's method and its variants by relaxing the assumption of Lipschitz continuity of the derivative of the function, which define the nonlinear equation under consideration; see [1–10].

In [4,9], under a majorant condition and generalized Lipschitz condition, respectively, local convergence, quadratic rate and estimate of the best possible convergence radius of Newton's method as well as uniqueness of the solution for the nonlinear equation in question were established. In the analysis presented in [4], *convexity* of the derivative of the scalar majorant function was assumed and in [9] the *nondecrement* of the positive integrable function which defines the generalized Lipschitz condition was assumed. These assumptions seem to be actually natural in the local analysis of Newton's method. The convergence, uniqueness, superlinear rate and estimate of the best possible convergence radius will be established in this paper without assuming the convexity of the derivative of the majorant function or that the function which defines the generalized Lipschitz condition is nondecreasing. In particular, this analysis shows that the convexity of the derivative of the majorant function or that the function which defines the generalized Lipschitz condition is nondecreasing is needed only to obtain quadratic convergence rate of the sequence generated by Newton's Method. Also, as in [4], the analysis presented provides a clear relationship between the majorant function with the nonlinear operator under consideration. Besides improving the convergence theory this analysis permits us to obtain two new important special cases, namely, [7,10] (see also, [4,9]) as applications. It is worth pointing out that the majorant condition used here is equivalent to Wang's condition (see [10]) that the derivative of the majorant function is always convex.

The organization of the paper is as follows. In Section 1.1, some notations and one basic result used in the paper are presented. In Section 2, the main result is stated and in Section 2.1 some properties of the majorant function are established and the main relationships between the majorant function and the nonlinear operator used in the paper are presented. In Section 2.2, the uniqueness of the solution and the optimal convergence radius are obtained. In Section 2.3 the main result is proved and two applications of this result are given in Section 3. Some final remarks are given in Section 4.

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### 1.1. Notation and auxiliary results

The following notations and results are used throughout our presentation. Let $X, Y$ be Banach spaces. The open and closed balls at $x$ are denoted, respectively, by

$$B(x, \delta) = \{y \in X;\ \|x - y\| < \delta\} \quad \text{and} \quad B[x, \delta] = \{y \in X;\ \|x - y\| \leqslant \delta\}.$$

Let $\Omega \subseteq X$ be an open set. The Fréchet derivative of $F : \Omega \to Y$ is the linear map $F'(x) : X \to Y$.

**Lemma 1** (Banach's Lemma)**.** *Let $B : X \to X$ be a bounded linear operator. If $I : X \to X$ is the identity operator and $\|B - I\| < 1$, then $B$ is invertible and $\|B^{-1}\| \le 1/(1 - \|B - I\|)$.*

## 2. Local analysis for Newton's method

Our goal is to state and prove a local theorem for Newton's method, which generalizes Theorem 2.1 of [4]. First, we will prove some results regarding the scalar majorant function, which relaxes the Lipschitz condition. Then we will establish the main relationships between the majorant function and the nonlinear function. We will also prove the uniqueness of the solution in a suitable region and the optimal ball of convergence. Finally, we will show the well definedness of Newton's method and convergence, also results on the convergence rates will be given. The statement of the theorem is:

**Theorem 2.** *Let $X$, $Y$ be Banach spaces, $\Omega \subseteq X$ be an open set and $F : \Omega \to Y$ be a continuously differentiable function. Let $x_* \in \Omega$, $R > 0$ and $\kappa := \sup\{t \in [0, R) : B(x_*, t) \subset \Omega\}$. Suppose that $F(x_*) = 0$, $F'(x_*)$ is invertible and there exists an $f : [0, R) \to \mathbb{R}$ continuously differentiable such that*

$$\left\| F'(x_*)^{-1} \left[ F'(x) - F'(x_* + \tau(x - x_*)) \right] \right\| \le f'(\|x - x_*\|) - f'(\tau \|x - x_*\|), \tag{1}$$

*for all $\tau \in [0, 1]$, $x \in B(x_*, \kappa)$ and*

(h1) $f(0) = 0$ *and* $f'(0) = -1$;
(h2) $f'$ *is strictly increasing.*

*Let $\nu := \sup\{t \in [0, R) : f'(t) < 0\}$, $\rho := \sup\{\delta \in (0, \nu) : [f(t)/f'(t) - t]/t < 1,\ t \in (0, \delta)\}$ and*

$$r := \min\{\kappa, \rho\}.$$

*Then the sequences with starting points $x_0 \in B(x_*, r)/\{x_*\}$ and $t_0 = \|x_0 - x_*\|$, respectively, namely*

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \qquad t_{k+1} = |t_k - f(t_k)/f'(t_k)|, \quad k = 0, 1, \ldots, \tag{2}$$

*are well defined; $\{t_k\}$ is strictly decreasing, is contained in $(0, r)$ and converges to 0 and $\{x_k\}$ is contained in $B(x_*, r)$ and converges to the point $x_*$ which is the unique zero of $F$ in $B(x_*, \sigma)$, where $\sigma := \sup\{t \in (0, \kappa) : f(t) < 0\}$ and there hold:*

$$\lim_{k\to\infty} [\|x_{k+1} - x_*\| / \|x_k - x_*\|] = 0, \qquad \lim_{k\to\infty} [t_{k+1}/t_k] = 0. \tag{3}$$

*Moreover, if $f(\rho)/(\rho f'(\rho) - 1) = 1$ and $\rho < \kappa$ then $r = \rho$ is the best possible convergence radius.*
*If, additionally, given $0 \le p \le 1$*

(h3) *the function $(0, \nu) \ni t \mapsto [f(t)/f'(t) - t]/t^{p+1}$ is strictly increasing,*

*then the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing and there holds*

$$\|x_{k+1} - x_*\| \le \left[ t_{k+1}/t_k^{p+1} \right] \|x_k - x_*\|^{p+1}, \quad k = 0, 1, \ldots. \tag{4}$$

**Remark 1.** The first equation in (3) means that $\{x_k\}$ converges superlinearly to $x_*$. Moreover, because the sequence $\{t_{k+1}/t_k^{p+1}\}$ is strictly decreasing $t_{k+1}/t_k^{p+1} \le t_1/t_0^{p+1}$, for $k = 0, 1, \ldots$. So, the inequality in (4) implies $\|x_{k+1} - x_*\| \le [t_1/t_0^{p+1}]\|x_k - x_*\|^{p+1}$, for $k = 0, 1, \ldots$. As a consequence, if $p = 0$ then $\|x_k - x_*\| \le t_0[t_1/t_0]^k$ for $k = 0, 1, \ldots$ and if $0 < p \le 1$ then

$$\|x_k - x_*\| \le t_0 \, (t_1/t_0)^{[(p+1)^k - 1]/p}, \quad k = 0, 1, \ldots.$$

**Example 1.** The following continuously differentiable functions satisfy (h1), (h2) and (h3):

(i) $f : [0, +\infty) \to \mathbb{R}$ such that $f(t) = t^{1+p} - t$;
(ii) $f : [0, +\infty) \to \mathbb{R}$ such that $f(t) = \mathrm{e}^{-t} + t^2 - 1$.

Letting $0 < p < 1$, the derivative of first function as well as that of the second is not convex.