Journal of Computational and Applied Mathematics 235 (2011) 1531–1541

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Periodic and homoclinic solutions generated by impulses for asymptotically linear and sublinear Hamiltonian system

Xizhi Han *, Hao Zhang

Department of Mathematics and Systems Science, College of Science, National University of Defense Technology, Changsha 410073, PR China

ARTICLE INFO

Article history:
Received 23 June 2009
Received in revised form 28 August 2010

MSC:
34A37
34B37
34C25
34C37

Keywords:
Impulsive differential equation
Periodic solution
Homoclinic solution
Critical point

ABSTRACT

In this paper, we get the existence of periodic and homoclinic solutions for a class of asymptotically linear or sublinear Hamiltonian systems with impulsive conditions via variational methods. However, without impulses, there is no homoclinic or periodic solution for the system considered in this paper. Moreover, our results can be used to study the existence of periodic and homoclinic solutions of difference equations.

0. Introduction

In this paper, we consider the following Hamiltonian systems with impulsive conditions

$$\ddot{q}(t) = f(t, q(t)), \quad \text{for } t \in (s_{k-1}, s_k), \tag{0.1}$$

$$\Delta\dot{q}(s_k) = g_k(q(s_k)), \tag{0.2}$$

where $k \in \mathbb{Z}$, $q \in \mathbb{R}^n$, $\Delta\dot{q}(s_k) = \dot{q}(s_k^+) - \dot{q}(s_k^-)$ with $\dot{q}(s_k^\pm) = \lim_{t\to s_k^\pm} \dot{q}(t)$, $f(t, q) = \operatorname{grad}_q F(t, q)$, $F(t, q) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $g_k(q) = \operatorname{grad}_q G_k(q)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for each $k \in \mathbb{Z}$ and there exist $m \in \mathbb{N}$ and $T \in \mathbb{R}^+$ such that $0 = s_0 < s_1 < \cdots < s_m = T$, $s_{k+m} = s_k + T$ and $g_{k+m} = g_k$ for all $k \in \mathbb{Z}$.

There have been plenty of works studying the existence of periodic and homoclinic solutions of ordinary differential equations

$$\ddot{q}(t) = f(t, q(t)), \quad \text{for } t \in \mathbb{R}, \tag{0.3}$$

for examples, see [1–4]. In [2], Rabinowitz put forward the well known Ambrosetti–Rabinowitz (AR) condition: there exist some $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(t, x) \le f(t, x)x \tag{0.4}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$ with $|x| \ge R$ and proved the existence of periodic solutions for system (0.3). Under the (AR) condition, the action functional associated with system (0.3) satisfies the Palais–Small condition ((PS) condition). Since then, it has been used extensively to study the existence of solutions of the superlinear Hamiltonian system, for examples, see [5,6].

* Corresponding author.
E-mail addresses: hanxizhi@gmail.com (X. Han), zhmath@gmail.com (H. Zhang).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.08.040

However, the (PS) condition does not hold when $f(t,x)$ is asymptotically linear or sublinear, which brings some difficulties for system (0.3). In literature, more works concentrated on the first-order Hamiltonian systems which are resonant at infinity, for examples, see [7,8]. In [9], Bartolo and Benci studied the existence of solutions for a second-order Hamiltonian system. They proved a deformation lemma by substituting the usual (PS) condition by a (C)-condition and it turns out that the Mountain Pass Theorem holds under the (C)-condition. By the same method, the authors in [10] studied the existence of periodic solutions of system (0.3) when the behavior of the gradient of nonlinearity at infinity is like that of a real valued function $L_\infty(t)x$.

Motivated by the works mentioned above, in this paper we study the existence of homoclinic and periodic solutions of the second-order impulsive system (0.1)–(0.2) by variational methods. To the best of our knowledge, there has not been any result of system (0.1)–(0.2) when $f(t,q)$ is asymptotically linear or sublinear.

Impulsive differential equations are used to describe the behavior of evolution processes in which their states change abruptly at certain moments of time. These kind of processes naturally occur in control theory, biology, optimization theory, medicine and some physics and mechanics problems, for examples, see [11–15]. For general aspects of impulsive differential equations, see [16–19] and references therein.

The classical methods to study impulsive differential equations are mainly the topological degree theory [20,21] and comparison method [22,23]. Recently, some authors studied these equations via variational methods [24–28]. In [24], Tian and Ge studied the existence of solutions for a second order impulsive system of a more general form than (0.1)–(0.2) with Sturm–Liouville boundary conditions. In [29,28], the authors studied the existence of weak solutions for a system similar to (0.1)–(0.2) with Dirichlet boundary conditions. In [26,27], Zhang and Li studied the existence of periodic solutions for systems (0.1)–(0.2).

In this paper, we study the existence of periodic and homoclinic solutions for (0.1)–(0.2) when the nonlinear term does not satisfy the (AR) condition, or more precisely, when $f(t,x)$ is asymptotically linear or sublinear. Particularly, the effect of impulses are stressed and some results owned by impulsive differential equation are attained. We first get a periodic solution of (0.1)–(0.2). Next, when $f(t,x)$ is asymptotically linear, we show there exists a non-zero homoclinic solution as the limit of a sequence $2lT$-periodic solutions as l goes to ∞, and when $f(t,x)$ is sublinear, we attain a non-zero homoclinic solution in the same way by strengthening the effect of impulses. Under our conditions about f, system (0.3) does not possess any non-zero periodic or homoclinic solutions. So the periodic and homoclinic solutions we obtained are solutions generated by impulses. Here, a periodic (or homoclinic) solution q of (0.1)–(0.2) is said to be generated by impulses if system (0.3) does not possess periodic (or homoclinic) solutions. Particularly, in our results, the nonlinear term is allowed to be zero. Using these facts, we apply these results to a class of difference equations, and obtain two new results about the existence of periodic and homoclinic solutions.

This paper is divided into five sections: in Section 1, the main results are stated; in Section 2, some preliminaries are given; in Sections 3 and 4, we give the proofs of the main results; in Section 5, we apply our results to difference equations.

1. Main results

First, we give the following conditions:

(f_1) $F(t,q) \geq \frac{1}{2}f(t,q)q > 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n \setminus \{0\}$.

(f_2) $f(t,q) = \alpha q + w(t,q)$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n$, where $\alpha > \frac{2}{\mu}$ for some $\mu > 2$, $w(t,q) = \operatorname{grad}_q W(t,q)$ and $W(t,q) \geq \frac{1}{2}w(t,q)q > 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n \setminus \{0\}$;

(f_3) $F(t,q) \geq \frac{1}{2}f(t,q)q \geq 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n$.

(g_1) there exists $\mu > 2$ such that

$$g_k(q)q \leq \mu G_k(q) < 0, \quad \text{for all } k = 1,2,\ldots,m \text{ and } q \in \mathbb{R}^n \setminus \{0\}.$$

(g_2) $g_k(q) = 2q + w_k(q)$, where $w_k = \operatorname{grad}_q W_k(q)$ and satisfies that there exists $\mu > 2$ such that

$$w_k(q)q \leq \mu W_k(q) < 0, \quad \text{for all } k = 1,2,\ldots,m \text{ and } q \in \mathbb{R}^n \setminus \{0\}.$$

Theorem 1. *If F is T-periodic in t and satisfies (f_1), g_ks satisfies condition (g_1) for all $k = 1,2,\ldots,m$, then system (0.1)–(0.2) possesses at least one non-zero periodic solution generated by impulses.*

Example 1. Let

$$F(t,q) = \frac{1}{4}e(t)q^{\frac{4}{3}},$$

where $e : \mathbb{R} \to \mathbb{R}^+$ is continuous with period T. Then $f(t,q) = \frac{1}{3}e(t)q^{\frac{1}{3}}$. It is easy to check that $f(t,q)$ satisfies condition (f_1). Moreover, we take the impulses G_k ($k = 1,\ldots,m$) as follows

$$G_k(q) = -\eta(k)q^4, \tag{1.1}$$