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Pricing and hedging of financial derivatives using a posteriori error estimates and adaptive methods for stochastic differential equations

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A B S T R A C T

The efficient and accurate calculation of sensitivities of the price of financial derivatives with respect to perturbations of the parameters in the underlying model, the so-called 'Greeks', remains a great practical challenge in the derivative industry. This is true regardless of whether methods for partial differential equations or stochastic differential equations (Monte Carlo techniques) are being used. The computation of the 'Greeks' is essential to risk management and to the hedging of financial derivatives and typically requires substantially more computing time as compared to simply pricing the derivatives. Any numerical algorithm (Monte Carlo algorithm) for stochastic differential equations produces a time-discretization error and a statistical error in the process of pricing financial derivatives and calculating the associated 'Greeks'. In this article we show how a posteriori error estimates and adaptive methods for stochastic differential equations can be used to control both these errors in the context of pricing and hedging of financial derivatives. In particular, we derive expansions, with leading order terms which are computable in a posteriori form, of the time-discretization errors for the price and the associated 'Greeks'. These expansions allow the user to simultaneously first control the time-discretization errors in an adaptive fashion, when calculating the price, sensitivities and hedging parameters with respect to a large number of parameters, and then subsequently to ensure that the total errors are, with prescribed probability, within tolerance.

1. Introduction

It is fair to say that it is still a great practical challenge in the derivative industry to efficiently and accurately calculate the so-called 'Greeks', that is sensitivities of the price of financial derivatives with respect to perturbations of the parameters in the underlying model. Focusing on methods based on stochastic differential equations, the calculation of these sensitivities remains a particularly topical area of current research and the prevailing techniques include finite difference approximations, pathwise derivative estimates, the likelihood ratio method and its generalizations using the Malliavin calculus. We refer to [1] for an excellent account of these methods and their advantages and disadvantages. Although [2,1,3] contain most of the relevant references on these topics we here still would like to suggest [4–14] as additional references for the interested reader. We emphasize that while these articles are almost exclusively devoted to financial applications, the techniques developed are also useful in many other contexts. Moreover, we note that a key feature of the techniques in many of these articles is, heuristically, that the computations tend to be organized in a forward looking way where the calculations in the next step depend on the calculations up to the present. However, in [3] an adjoint formulation for the calculation of sensitivities is suggested and it is shown, numerically, that this formulation can be used to accelerate the calculation of

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the 'Greeks'. The method outlined in [3] is particularly well suited in applications requiring sensitivities to a large number of parameters and particular examples of such applications include interest rate derivatives requiring sensitivities with respect to all initial forward rates and equity derivatives requiring sensitivities with respect to all points on a volatility surface. Furthermore, as emphasized in [3] the adjoint method has its advantages, compared to competing methods with forward looking features, when calculating the sensitivities of a small number of securities with respect to a large number of parameters. On the contrary, competing methods with forward looking features are advantageous when calculating the sensitivities of many securities with respect to a small number of parameters. The notion of 'small number of securities' can here be an entire book, consisting of many individual securities, as long as the sensitivities to be calculated are for the book as a whole and not for the constituent securities.

In this article we further develop the adjoint method suggested in [3] by outlining how a posteriori error estimates and adaptive methods for stochastic differential equations can be used to adaptively first control the time-discretization errors in these calculations and then to ensure that the total error, defined as sum of the time-discretization error and the statistical error, is, with prescribed probability, within tolerance. In particular, we give a theoretically sound base for the adjoint method suggested in [3]. Our results concerning a posteriori error estimates and adaptive methods for stochastic differential equations build and expand on the work by Szepessy et al. [15] concerning adaptive weak approximations of stochastic differential equations and, to our knowledge, a posteriori error estimates for stochastic differential equations applied to the pricing of financial derivatives and, in particular, applied to the calculation of hedging parameters for financial derivatives, have previously not been discussed in the literature. Hence, we claim to give a novel contribution to the literature concerning the numerical aspects of pricing and hedging of financial derivatives, as well as to the general problem of conducting sensitivity analysis for solutions of second order parabolic partial differential equations using stochastic techniques. Finally, this article is based on the results developed in the thesis of the second author, see [16].

To more thoroughly describe the methodology outlined in this article we first have to introduce some notation. Let $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n$ and let $M(n, \mathbb{R})$ be the set of all $n \times n$-matrices with real valued entries. Given a matrix $\sigma \in M(n, \mathbb{R})$ its transpose is denoted by σ^*. Let

$$\begin{aligned} \mu(t, x) &= \mu(t, x, \theta_\mu) = \bar{\mu}(t, x) + \theta_\mu \tilde{\mu}(t, x), \\ \sigma(t, x) &= \sigma(t, x, \theta_\sigma) = \bar{\sigma}(t, x) + \theta_\sigma \tilde{\sigma}(t, x), \end{aligned} \tag{1.1}$$

where $\bar{\mu}, \tilde{\mu} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\bar{\sigma}, \tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}^n \to M(n, \mathbb{R})$, $\theta_\mu \in \mathbb{R}$, $\theta_\sigma \in \mathbb{R}$, and $|\theta_\mu| \leq \epsilon$, $|\theta_\sigma| \leq \epsilon$, for some small $\epsilon > 0$. $\tilde{\mu}$ and $\tilde{\sigma}$ represent perturbations of $\bar{\mu}$ and $\bar{\sigma}$. In the following we assume that there exists $\eta > 0$ such that the following ellipticity condition is satisfied,

$$\xi^*(\bar{\sigma}(t, x) + \theta_\sigma \tilde{\sigma}(t, x))(\bar{\sigma}(t, x) + \theta_\sigma \tilde{\sigma}(t, x))^* \xi \geq \eta |\xi|^2, \tag{1.2}$$

whenever $|\theta_\sigma| \leq \epsilon$, $\xi \in \mathbb{R}^n$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. The ellipticity condition in (1.2) is not crucial to the analysis outlined in this article. In fact, the more general assumption of hypoellipticity suffices as discussed at the end of the article. Define $\theta = (\theta_\mu, \theta_\sigma)$ and let, for $i \in \{1, \ldots, n\}$,

$$X_i(t) = X_i(t, \theta) = x_i + \int_0^t \mu_i(s, X(s), \theta) \mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma_{ij}(s, X(s), \theta) \mathrm{d}W_j(s). \tag{1.3}$$

Let $X(t) = (X_1(t), \ldots, X_n(t))^*$ denote the corresponding vector. Here $(W(t))_{0 \leq t \leq T}$, $W(t) = (W_1(t), \ldots, W_n(t))^*$, is a standard Brownian motion in $\mathbb{R}^n$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ with the usual assumptions on $(\mathcal{F}_t)_{0 \leq t \leq T}$. By a standard Brownian motion in $\mathbb{R}^n$ we mean a process whose components are independent one-dimensional Brownian motions. In the definition of $X_i(t) = X_i(t, \theta)$ we have indicated the dependence on the parameter vector $\theta = (\theta_\mu, \theta_\sigma)$. Assuming appropriate growth and regularity conditions on the coefficients μ_i and σ_{ij}, as will be discussed in detail below, the system in (1.3) has a unique strong solution for all parameters $\theta = (\theta_\mu, \theta_\sigma)$, $|\theta_\mu| \leq \epsilon$, $|\theta_\sigma| \leq \epsilon$. We recall that there is a well-known close connection between stochastic differential equations and second order parabolic partial differential equations. We therefore introduce the second order parabolic operator

$$L = \frac{1}{2} \sum_{i,j=1}^n [\sigma \sigma^*]_{ij}(t, x) \partial_{ij} + \sum_{i=1}^n \mu_i(t, x) \partial_i, \tag{1.4}$$

and we note that the structural assumption on the operator L, imposed by (1.2), is that the operator $\partial_t + L$ is uniformly elliptic–parabolic. Let $T > 0$ and let the function $g : \mathbb{R}^n \to \mathbb{R}$ be given. Define

$$u(t, x) = u(t, x, \theta) = u(t, x, (\theta_\mu, \theta_\sigma)) = E[g(X(T, \theta)) | X(t, \theta) = x]. \tag{1.5}$$

Then, under appropriate smoothness and growth conditions on μ_i, σ_{ij} and g, the Feynman–Kac formula asserts that u in (1.5) is the unique solution to the Cauchy problem

$$\begin{cases} \partial_t u(t, x) + Lu(t, x) = 0, & \text{whenever } (t, x) \in (0, T) \times \mathbb{R}^n, \\ u(T, x) = g(x), & \text{whenever } x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$