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# A characteristic finite element method for optimal control problems governed by convection–diffusion equations

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ABSTRACT

In this paper we analyze a characteristic finite element approximation of convex optimal control problems governed by linear convection-dominated diffusion equations with pointwise inequality constraints on the control variable, where the state and co-state variables are discretized by piecewise linear continuous functions and the control variable is approximated by either piecewise constant functions or piecewise linear discontinuous functions. A priori error estimates are derived for the state, co-state and the control. Numerical examples are given to show the efficiency of the characteristic finite element method.

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## 1. Introduction

Optimal control problems governed by convection–diffusion equations arise in many scientific and engineering applications, such as atmospheric pollution control problems [1,2]. Efficient numerical methods are essential to successful applications of such optimal control problems. To the best of my knowledge, there are only a few published results on optimal control problems governed by steady convection–diffusion equations; see [3] of SUPG method, [4] of standard finite element discretizations with stabilization based on local projection method, [5] of symmetric stabilization method; [6] of edge-stabilization method and [7] of the application of RT mixed DG scheme. For the approximation of constrained optimal control problems governed by time-dependent convection–diffusion equations, it is much more complicated and there are nearly no related papers published so far. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [8–13].

In this paper we consider the following linear-quadratic optimal control problems for the state variable $y$ and the control variable $u$:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\|y-z_d\|_{0,\Omega}^2 + \alpha\|u\|_{0,\Omega_U}^2\right) \mathrm{d}t, \tag{1.1}$$

subject to

$$\begin{cases} y_t + \mathbf{v}\cdot\nabla y - \operatorname{div}(A\nabla y) = f + Bu, & (x,t)\in\Omega\times(0,T],\\ y(x,t)=0, & (x,t)\in\partial\Omega\times[0,T],\\ y(x,0)=y_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

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and

$$\xi_1 \le u(x,t) \le \xi_2, \quad (x,t) \in \Omega_U \times (0,T], \tag{1.3}$$

where $\mathbf{v} = \mathbf{v}(x,t)$ denotes a velocity field in the flow control, $A = A(x,t)$ is a diffusion coefficient, $f = f(x,t)$ accounts for external sources and sinks, $B$ is a linear continuous operator, and $y_0(x)$ is a prescribed initial data. In our case, we assume that the convection term dominates the diffusion term. A precise formulation of this problem including a functional analytic setting is given in the next section.

The methods of characteristics [14–16] combine the convection and capacity terms in the governing equations to carry out the temporal discretization in a Lagrange coordinate. These methods symmetrize the governing equation and stabilize their numerical approximations. They generate accurate numerical solutions and significantly reduce the numerical diffusion and grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. The goal of the present paper is to apply the methods of characteristics to the quadratic optimal control problems governed by linear convection-dominated diffusion equations, and we obtain a priori error estimates for both the control and state approximations. The present paper extends [17] in two aspects: First, it deals with either piecewise linear elements or piecewise constant elements for the control approximation. Second, the error estimates are obtained in the framework of $L^2$-error and bilateral pointwise inequality control constraints. The results obtained and the techniques used here are also different from that of [17].

The rest of the paper is organized as follows: In Section 2, we first refine the statement of the model problem and then derive a generic weak formulation and optimality conditions. In Section 3, we construct a characteristic finite element approximation scheme for the optimal control problems. In Section 4, the main error estimates are derived for the control problems with obstacle constraints. In Section 5, we conduct some numerical experiments to observe the convergence behavior of the numerical scheme. Section 6 contains concluding remarks.

In this paper, we denote $C$ and $\delta$ be a generic constant and small positive number which are independent of the discrete parameters and may have different values in different circumstances, respectively.

## 2. Optimal control problems and optimality conditions

Let $\Omega$ and $\Omega_U$ be bounded open sets in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. Just for simplicity of presentation, we assume that $\Omega$ and $\Omega_U$ are convex polygon. We employ the usual notion for Lebesgue and Sobolev spaces; see [8,9] for details.

Now we give a description of the mathematical model of the optimal control problems governed by convection–diffusion equations. To fix the idea, let $I = (0,T]$ and we shall take the state space $W = H^1(I;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(I;U)$ with $U = L^2(\Omega_U)$, and the observation space $Y = L^2(I;H)$ with $H = L^2(\Omega)$. $B$ is a linear continuous operator from $U$ to $H$, and $K$ is a closed convex set in $X$.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_T} = T$ be a subdivision of $I$, with corresponding time intervals $I_n = (t_{n-1}, t_n]$ and time steps $k_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, N_T$. Denote $k = \max_{1\le n\le N_T} k_n$ and $f^n = f(t_n)$. We define, for $1 \le q < \infty$, the discrete time-dependent norms

$$\|f\|_{l^q(I;X)} = \left(\sum_{n=1}^{N_T} k_n \|f^n\|_X^q\right)^{\frac{1}{q}}$$

and the standard modification for $q = \infty$. Let

$$l^q(I;X) := \left\{f : \|f\|_{l^q(I;X)} < \infty\right\}, \quad 1 \le q \le \infty.$$

In problems (1.1)–(1.3), $\alpha$ is a positive constant, the bounds $\xi_1, \xi_2$ are two real numbers that fulfill $\xi_1 < \xi_2$, $f \in L^2(I;L^2(\Omega))$, $z_d \in H^1(I;L^2(\Omega))$, $y_0 \in V = H_0^1(\Omega)$, and

$$A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\bar{\Omega}))^{2\times 2},$$

such that there is a positive constant $c$ satisfying

$$\sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \ge c|\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

The velocity field vector $\mathbf{v} = (V_1(x,t), V_2(x,t))$ lies in the function space $L^\infty(I;W^{1,\infty}(\bar{\Omega})^2)$ and is divergence-free, i.e.,

$$\nabla \cdot \mathbf{v} = 0, \quad \forall x \in \Omega,\ t \in I.$$

To avoid technical boundary difficulties associate with the methods of characteristics, we assume that $\Omega$ is a rectangle and the state equation is $\Omega$-periodic, i.e., we assume that all functions in Eq. (1.2) are spatially $\Omega$-periodic; see, [14,15] for example.

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