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Maximum norm error estimates of the Crank–Nicolson scheme for solving a linear moving boundary problem[☆]

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ABSTRACT

The Crank–Nicolson scheme is considered for solving a linear convection–diffusion equation with moving boundaries. The original problem is transformed into an equivalent system defined on a rectangular region by a linear transformation. Using energy techniques we show that the numerical solutions of the Crank–Nicolson scheme are unconditionally stable and convergent in the maximum norm. Numerical experiments are presented to support our theoretical results.

1. Introduction

Moving boundary problems occur in the mathematical modelling of many physical processes involving diffusion, such as the movement of the shoreline in a sedimentary ocean [1], the drift and collection of oil [2], heat conduction across the solid from a liquid–solid interface to the cooled surface [3] etc. Moving boundary problems also exist in the swelling of biological tissues [4,5] and the swelling of polymers [6].

Due to the difficulties in obtaining analytical solutions, it is important to develop numerical methods for moving boundary problems. Recently, more finite difference schemes have been used for dealing with moving boundary problems [7–10], but there are no analyses of the convergence and stability of difference schemes. In addition, Baines and Hubbard [11] established a moving mesh finite element algorithm for moving boundary problems. Immersed interface methods and immersed boundary methods also have been used to deal with moving boundary problems [12–14].

The linear convection–diffusion equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(au-\kappa\frac{\partial u}{\partial x}\right)=g(x,t),\quad (x,t)\in Q_T \tag{1.1}$$

along with the initial value condition

$$u(x,0)=u_0(x),\quad x\in\Omega_0, \tag{1.2}$$

and the moving boundary value conditions

$$u(x,t)=\Phi(x,t),\quad x\in\partial\Omega_t,\ 0<t\le T \tag{1.3}$$

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as a mathematical model is widely used in various applications, where $Q_T = \left\{(x,t) \in R^2, x \in \Omega_t, t \in [0,T]\right\}$, Ω_t is an interval in R for each $t \in [0,T]$, a and κ are two positive constants.

Moving mesh method is very effective for dealing with moving boundary problems. Mackenzie and Mekwi [15] discussed the stability and convergence of time integration schemes for the solution of (1.1)–(1.3) (they took $g(x,t) = \Phi(x,t) = 0$). Using variable mesh method and energy techniques they showed that the backward Euler scheme is unconditionally stable in a mesh-dependent L_2 norm, but the Crank–Nicolson scheme is only conditionally stable.

Sun [2] gave a three-level linearized and weak coupled difference scheme using a moving mesh for the model of oil drift and collection with moving boundary value, and analyzed the solvability and convergence of the difference scheme. He proved that the convergence order of the difference scheme is $O(\tau^2 + h^2)$.

In this article, the Crank–Nicolson scheme for the linear convection–diffusion equation with moving boundaries (1.1)–(1.3) is analyzed. A linear transformation is introduced in our analysis to transform (1.1) to an equivalent equation defined on a rectangle region. It is proved that the Crank–Nicolson scheme for (1.1)–(1.3) is unconditionally stable and convergent in the maximum norm. The convergence order is $O(\tau^2 + h^2)$.

The contents will be organized as follows. In the next section, an equivalent system defined on a rectangular region is achieved by making a linear transformation to Eq. (1.1). Mesh generation and some notations are also introduced in this section. The Crank–Nicolson scheme is constructed for the equivalent system in Section 3. Section 4 presents the energy analysis for the Crank–Nicolson scheme and gives the main results of the article. Numerical experiments are provided to support our theoretical results in Section 5.

2. A linear transformation and mesh generation

Assume that the initial value u_0 and exterior force g are regular enough in (1.1)–(1.3), the boundary value Φ is piecewise smooth, $u_0(x) = \Phi(x,0)$, $x \in \partial\Omega_0$, and the domain Ω_t can be defined as $\Omega_t = [x_l(t), x_r(t)]$, where the functions $x_l(t), x_r(t) \in C^1[0,T]$, and $x_l(t) < x_r(t)$ for every $t \in [0,T]$.

Introduce a linear transformation

$$\begin{cases} x = (1-\xi)x_l(t) + \xi x_r(t), & 0 \le \xi \le 1 \\ t = t, & 0 \le t \le T \end{cases} \tag{2.1}$$

and denote $w(\xi,t) = u\left((1-\xi)x_l(t) + \xi x_r(t), t\right)$, $G(\xi,t) = g\left((1-\xi)x_l(t) + \xi x_r(t), t\right)$. Then we have

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}, \qquad \frac{\partial w}{\partial \xi} = \frac{\partial u}{\partial x}x_\xi(t), \\ &\frac{\partial^2 w}{\partial \xi^2} = \frac{\partial u}{\partial x}x_{\xi\xi}(t) + \frac{\partial^2 u}{\partial x^2}x_\xi^2(t). \end{aligned} \tag{2.2}$$

It is obvious that

$$x_\xi(t) = x_r(t) - x_l(t) > 0, \qquad x_{\xi\xi}(t) = 0. \tag{2.3}$$

Performing in (1.1)–(1.3) the substitution (2.1) and then using (2.2)–(2.3) we obtain

$$\frac{\partial w}{\partial t} - \frac{\kappa}{(x_\xi)^2}\frac{\partial^2 w}{\partial \xi^2} - \frac{1}{x_\xi}\left(\frac{\partial x}{\partial t} - a\right)\frac{\partial w}{\partial \xi} = G(\xi,t), \quad (\xi,t) \in Q_R, \tag{2.4}$$

along with the initial value condition

$$w(\xi,0) = w_0(\xi), \quad 0 \le \xi \le 1 \tag{2.5}$$

and the boundary value conditions

$$w(0,t) = \phi_1(t), \qquad w(1,t) = \phi_2(t), \quad 0 < t \le T \tag{2.6}$$

where $Q_R = \left\{(\xi,t) \in R^2,\ 0 \le \xi \le 1, 0 < t < T\right\}$, $w_0(\xi) = u_0\left((1-\xi)x_l(0) + \xi x_r(0)\right)$, $\phi_1(t) = \Phi\left(x_l(t), t\right)$, $\phi_2(t) = \Phi\left(x_r(t), t\right)$.

Summarizing above results, we obtain the following theorem.

Theorem 1. *Assume that the interval Ω_t can be defined as $\Omega_t = [x_l(t), x_r(t)]$, where the functions $x_l(t), x_r(t) \in C^1[0,T]$. If $x_l(t) < x_r(t)$ for every $t \in [0,T]$, then the problem (1.1)–(1.3) is equivalent to (2.4)–(2.6).*

Let

$$\Omega_h(t) \equiv \{x_i(t) \mid x_i(t) = ih(t),\ 0 \le i \le M\}$$

be a variable mesh of the interval $\Omega_t = [x_l(t), x_r(t)]$ with $h(t) = \frac{1}{M}[x_r(t) - x_l(t)]$, where $t \in [0,T]$ is fixed. Let

$$\widetilde{\Omega}_h \equiv \{\xi_i \mid \xi_i = ih,\ 0 \le i \le M\}$$