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Convergence of a FEM and two-grid algorithms for elliptic problems on disjoint domains

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ABSTRACT

In this paper, we analyze a FEM and two-grid FEM decoupling algorithms for elliptic problems on disjoint domains. First, we study the rate of convergence of the FEM and, in particular, we obtain a superconvergence result. Then with proposed algorithms, the solution of the multi-component domain problem (simple example — two disjoint rectangles) on a fine grid is reduced to the solution of the original problem on a much coarser grid together with solution of several problems (each on a single-component domain) on fine meshes. The advantage is the computational cost although the resulting solution still achieves asymptotically optimal accuracy. Numerical experiments demonstrate the efficiency of the algorithms.

1. Introduction

In this paper, we consider coupling elliptic transmission problems on a multi-component domain with nonlocal interface conditions on parts of the boundary of the components. The study of such problems could be motivated physically by the occurrence of various nonstandard boundary and coupling conditions in modern physics, biology and engineering [1–3]. A typical physical example is the stationary problem for radiative–conductive heat transfer in a system on opaque bodies [1,2]. Mathematical results in this direction were first obtained in [3]. Later, many works were devoted to the solvability of such problems. A survey concerning the mathematical theory of heat transfer in conducting and radiating bodies can be found in [1].

Some one dimensional problems of this type was studied numerically in [4–7]. The two-grid method was proposed in [8,9], independently of each other, for a linearization of nonlinear elliptic problems. The two-grid finite element method was also used by Xu and many other scientists (see the reference in [10]) for discretizing nonsymmetric indefinite elliptic and parabolic equations. By employing two finite element spaces of different scales, one coarse and one fine space, the method was used for symmetrization of nonsymmetric problems, which reduces the solution of a nonsymmetric problem on a fine grid to the solution of a corresponding (but much smaller) nonsymmetric problem, discretized on the coarse grid and the solution of a symmetric positive definite problem on the fine grid. This method was also used for location and parallelization for solving a large class of partial differential equations [11,12]. There are many other authors who have used this method for many different applications; see [13–15] among others.

The two-grid approach in the present paper is an extension of the idea in [10], where it was used to decouple a Shrödinger system of differential equations. The system of partial differential equations is first discretized on the coarse grid, then a decoupled system is discretized on the fine mesh. As a result, the computational complexity of solving the Shrödinger system is comparable with solving two decoupled Poisson equations on the same grid.

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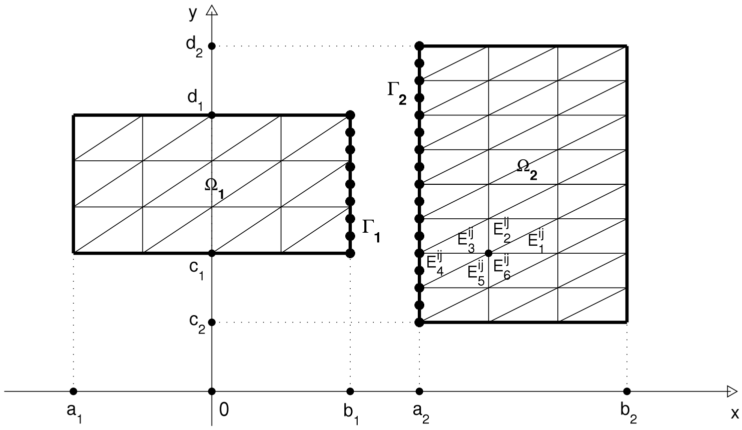

Fig. 1. Domains Ω_1 and Ω_2.

However, here we want to illustrate the two-grid idea in a new direction, *namely we use the two-grid discretization method to decouple the multi-component domain problem to several elliptic equations, each of them solved on its own domain*. For clarity, we use a simple model problem of linear elliptic equation on two disjoint rectangles.

In some industrial applications such as crystal growth, radiative heat transfer in materials that are conductive, gray and semitransparent, situations are relevant in which a transparent medium is enclosed by or several opaque, of diffusive gray bodies, for example *rectangles* in the 2D models in [2,16].

Also, our analysis should provide some insights on how a multiscale idea can be applied to multi-component domain problems when for example, some of the domains require very fine mesh. Finally, for nonlinear problems, iterative processes are often used to overcome the nonlinearities, see for example [17,4] and the resulting linear problems have the form (1)–(5).

In this paper we consider a system of two elliptic equations, each of them solved on its own rectangle. The two problems are coupled with nonlocal interface conditions on parts of the rectangles' boundary. First, the original system is approximated on the coarse mesh and then a decoupled system is discretized on the fine grid.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem, used to illustrate our method. In the next section, we discuss the convergence property of the FEM. Two-grid algorithms are proposed an analyzed in Section 4. In the last section we present results of numerical experiments, showing the effectiveness of our method.

In the next section, by C we denote a positive constant, independent of the boundary value solution and mesh sizes.

2. Two-rectangle model problem

As a model example we study the elliptic problem, defined on the disjoint rectangles $\Omega_n = (a_n, b_n) \times (c_n, d_n)$ with boundaries $\partial\Omega_n,\ n = 1, 2$, see Fig. 1,

$$-\frac{\partial}{\partial x}\left(p^n \frac{\partial u^n}{\partial x}\right) - \frac{\partial}{\partial y}\left(q^n \frac{\partial u^n}{\partial y}\right) + r^n u^n = f^n, \quad (x, y) \in \Omega_n, \tag{1}$$

$$u^n(x, c_n) = u^n(x, d_n) = 0, \quad a_n \le x \le b_n, \tag{2}$$

$$u^1(a_1, y) = 0, \quad c_1 \le y \le d_1, \qquad u^2(b_2, y) = 0, \quad c_2 \le y \le d_2, \tag{3}$$

$$p^1(b_1, y)\frac{\partial u^1}{\partial x}(b_1, y) + s^1(y)u^1(b_1, y) = \int_{c_2}^{d_2} \varphi^2(y, \eta)u^2(a_2, \eta)\mathrm{d}\eta, \quad c_1 \le y \le d_1, \tag{4}$$

$$-p^2(a_2, y)\frac{\partial u^2}{\partial x}(a_2, y) + s^2(y)u^2(a_2, y) = \int_{c_1}^{d_1} \varphi^1(y, \eta)u^1(b_1, \eta)\mathrm{d}\eta, \quad c_2 \le y \le d_2. \tag{5}$$

Throughout the paper we assume that the input datum satisfy the usual regularity and ellipticity conditions in Ω_n, $n = 1, 2$,

$$p^n, q^n, r^n \in L_\infty(\Omega_n), \tag{6}$$

$$0 < p_0^n \le p^n(x, y), \qquad 0 < q_0^n \le q^n(x, y), \tag{7}$$

$$s^n \in L_\infty(c_n, d_n), \qquad \varphi^n \in L_\infty((c_{3-n}, d_{3-n}) \times (c_n, d_n)). \tag{8}$$

In the real physical problems (see [1,2]) we often have $s^n, \varphi^n > 0,\ n = 1, 2$.