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Numerical analysis of a contact problem including bone remodeling ☆

J.R. Fernández *, R. Martínez

Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Escola Politécnica Superior, Campus Univ. s/n, 27002 Lugo, Spain

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ABSTRACT

In this work, a contact problem between an elastic body and a deformable obstacle is numerically studied. The bone remodeling of the material is also taken into account in the model and the contact is modeled using the normal compliance contact condition. The variational problem is written as a nonlinear variational equation for the displacement field, coupled with a first-order ordinary differential equation to describe the physiological process of bone remodeling. An existence and uniqueness result of weak solutions is stated. Then, fully discrete approximations are introduced based on the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. Error estimates are obtained, from which the linear convergence of the algorithm is derived under suitable regularity conditions. Finally, some 2D numerical results are presented to demonstrate the behavior of the solution.

1. Introduction

A contact problem between an elastic body and a deformable obstacle, including the bone remodeling process, is numerically studied in this paper. This bone remodeling model, derived by Cowin and Hegedus (see [1,2] and also the review paper [3]), is a generalization of the nonlinear elasticity, and it is based on the fact that the living bone is continuously adapting itself to external stimuli. Since this process has an enormous effect on the overall behavior and health of the entire body, the ability of these models to predict the bone remodeling is of great importance.

During the past ten years, some papers dealt with mathematical issues of these models as the existence and uniqueness of weak solutions under some quite strong assumptions (see, e.g., [4,5]), the analysis of an asymptotic rod model (see, for instance, [6]) or the numerical stability of finite element models (see [7]). Recently, other authors considered the fiber orientation and studied the energy dissipation associated to the bone remodeling (see, e.g., [8]). This paper extends the results presented in [9] to the case including contact and it continues the investigation reported in [10]. Here, our aim is to provide the numerical analysis of a fully discrete algorithm and to perform some 2D numerical simulations which demonstrate its behavior.

2. Mechanical and variational problems

Let us denote by $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, an open bounded domain and let $\Gamma = \partial\Omega$ be its outer surface which is assumed to be Lipschitz continuous and it is divided into three disjoint parts Γ_D, Γ_N and Γ_C such that meas $(\Gamma_D) > 0$. Let $[0, T]$, $T > 0$, be the time interval of interest. The body is being acted upon by a volume force of density $\boldsymbol{f}$, it is clamped on Γ_D and surface tractions with density $\boldsymbol{g}$ act on Γ_N. Finally, we assume that the body may come in contact with a deformable obstacle on the boundary part Γ_C which is located at a distance s, measured along the outward unit normal vector $\boldsymbol{\nu}$.

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* Corresponding author. Tel.: +34 981563100x23244; fax: +34 982285926.
E-mail address: jose.fernandez@usc.es (J.R. Fernández).

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Let $\boldsymbol{u} = (u_i)_{i=1}^d$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d$ be the displacement and the stress fields, respectively, let $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ represent the linearized strain field given by $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, and denote by e the so-called bone remodeling function, which measures the change in the volume fraction from a reference configuration.

The body is assumed elastic and, according to [1,2], the constitutive law is then written as $\boldsymbol{\sigma} = (\xi_0 + e)\mathcal{C}(e)\boldsymbol{\varepsilon}(\boldsymbol{u})$, where ξ_0 represents the reference volume fraction and $\mathcal{C}(e) = (C_{ijkl}(e))_{i,j,k,l=1}^d$ is a constitutive function whose properties will be described below.

Since the contact is assumed with a deformable obstacle, the well-known normal compliance contact condition is employed (see [11]); that is, the normal stress $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$ on Γ_C is given by $-\sigma_\nu = p_\nu(u_\nu - s)$, where $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ denotes the normal displacement in such a way that, when $u_\nu > s$, the difference $u_\nu - s$ represents the interpenetration of the body's asperities into those of the obstacle. The normal compliance function p_ν is prescribed and satisfies $p_\nu(r) = 0$ for $r \leq 0$, since then there is no contact. As an example, one may consider $p_\nu(r) = \mu\, r_+$, where $\mu > 0$ represents a deformability constant (that is, it denotes the stiffness of the obstacle), and $r_+ = \max\{0, r\}$. We also assume that the contact is frictionless, i.e. the tangential component of the stress field, denoted $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, vanishes on the contact surface.

Let us represent by $\cdot$ the inner product in $\mathbb{R}^d$ and by $|\cdot|$ its corresponding norm. Let $\mathbb{S}^d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d, and let: be its inner product and $|\cdot|$ its norm.

The evolution of the bone remodeling function is obtained from the first-order ordinary differential equation (see [1,2]) $\dot{e} = a(e) + \mathcal{A}(e) : \boldsymbol{\varepsilon}(\boldsymbol{u})$, where a dot above a variable represents the time derivative, $a(e)$ is a constitutive function and $\mathcal{A}(e) = (A_{ij}(e))_{i,j=1}^d$ denote the bone remodeling rate coefficients.

Let us define the truncation operator $\Phi_L : \mathbb{R} \to [-L, L]$ by $\Phi_L(r) = r$ if $|r| \leq L$, $\Phi_L(r) = L$ if $r > L$ and $\Phi_L(r) = -L$ if $r < -L$.

Finally, the process is assumed quasistatic and therefore, the inertia effects are neglected. Moreover, let e_0 denote the initial bone remodeling function.

The mechanical problem, derived from the continuum mechanics laws within the small displacement theory, is the following (see [2]).

Problem P. *Find the displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$, *the stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *and the bone remodeling function* $e : \Omega \times (0, T) \to \mathbb{R}$ *such that* $e(0) = e_0$ *and for a.e.* $t \in (0, T)$,

$$\boldsymbol{\sigma}(t) = (\xi_0 + e(t))\mathcal{C}(e(t))\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \text{in } \Omega,$$
$$\dot{e}(t) = a(e(t)) + \mathcal{A}(e(t)) : \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \text{in } \Omega,$$
$$-\mathrm{Div}\,\boldsymbol{\sigma}(t) = \gamma(\xi_0 + \Phi_L(e(t)))\boldsymbol{f}(t) \quad \text{in } \Omega,$$
$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_D,$$
$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{g}(t) \quad \text{on } \Gamma_N,$$
$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0}, \quad \sigma_\nu(t) = -p_\nu(u_\nu(t) - s) \quad \text{on } \Gamma_C.$$

Here, $\gamma > 0$ is the density of the full elastic material which is assumed constant for the sake of simplicity.

We turn now to obtain a variational formulation of Problem P. First, let us denote by $Y = L^2(\Omega)$ and $H = [L^2(\Omega)]^d$, and define the variational spaces $V = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$ and $Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d};\ \tau_{ij} = \tau_{ji},\ 1 \leq i, j \leq d\}$.

The following assumptions are done on the given data.

The elasticity coefficients $C_{ijkl}(e)$ are assumed to satisfy the following properties:

$$\left.\begin{array}{l}
\text{(a) There exists } L_C > 0 \text{ such that} \\
\qquad |(\xi_0 + e_1)C_{ijkl}(e_1) - (\xi_0 + e_2)C_{ijkl}(e_2)| \leq L_C|e_1 - e_2|, \ \ \forall e_1, e_2 \in \mathbb{R}. \\
\text{(b) There exists } M_C > 0 \text{ such that } |(\xi_0 + e)C_{ijkl}(e)| \leq M_C, \ \ \forall e \in \mathbb{R}. \\
\text{(c) } C_{ijkl}(e) = C_{jikl}(e) = C_{klij}(e) \text{ for } i, j, k, l = 1, \ldots, d. \\
\text{(d) There exists } m_C > 0 \text{ such that} \\
\qquad (\xi_0 + e)\mathcal{C}(e)\boldsymbol{\tau} : \boldsymbol{\tau} \geq m_C|\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d.
\end{array}\right\} \tag{1}$$

The normal compliance function $p_\nu : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}^+$ verifies:

$$\left.\begin{array}{l}
\text{(a) There exists } L_\nu > 0 \text{ such that} \\
\qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu\,|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{is Lebesgue measurable on } \Gamma_C, \ \forall r \in \mathbb{R}. \\
\text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)) \cdot (r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) = 0 \quad \text{for all } r \leq 0.
\end{array}\right\} \tag{2}$$