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# A two-step model for image denoising using a duality strategy and surface fitting[☆]

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ABSTRACT

In order to alleviate the staircase effect or the edge blurring in the course of the image denoising, we propose a two-step model based on the duality strategy. In fact, this strategy follows the observation that the dual variable of the restored image can be looked at as the normal vector. So we first obtain the dual variable and then reconstruct the image by fitting the dual variable. Following the augmented Lagrangian strategy, we propose a projection gradient method for solving this two-step model. We also give some convergence analyses of the proposed projection gradient method. Several numerical experiments are tested to compare our proposed model with the ROF model and the LLT model.



## 1. Introduction

Image denoising is a class of important and challenging inverse problems in image processing. The objective is to find the true image $u$ from an observed image $f$ defined by

$$f = u + \eta, \tag{1.1}$$

where $\eta$ is the additional noise. In order to solve such inverse problems, many approaches have been developed since the 1960s; see [1,2]. In these approaches, energy minimization based on the Tikhonov regularization has been demonstrated to be a powerful approach for tackling this kind of problem, which is defined by

$$\min \frac{\mu}{2}\|u - f\|^2_{L^2(\Omega)} + \Phi(u), \tag{1.2}$$

where $\mu$ is the regularization parameter, $\Omega$ is a bounded domain in $R^2$ with Lipschitz boundary and the regularization function $\Phi(u)$ usually satisfies the following assumptions [3]:

(A1) $\Phi : R^+ \to R^+$ and $\Phi(0) = \Phi'(0) = 0$,

(A2) $\Phi$ is sublinear at infinity, i.e., $\lim_{s\to+\infty} \frac{\Phi(s)}{s} = 0$.

The first condition means that, at the origin, the intensity variations of $\Phi(0)$ and $\Phi'(0)$ are weak. So we would like to encourage smoothing in all directions. In contrast, the second condition implies that the cost of edges is low, so the regularization function can preserve edges.

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One of the most famous regularization functions, the total variation (TV) seminorm, was first proposed in [4] in the image denoising problems. In [4], the following model (called the ROF model) was considered:

$$\min_{u\in BV(\Omega)} E_1(u) = \frac{\lambda}{2}\int_\Omega (f-u)^2 \mathrm{d}x + \int_\Omega |\nabla u| \mathrm{d}x, \tag{1.3}$$

where $BV(\Omega)$ denotes the space of functions with bounded variation on $\Omega$. Since the ROF model (1.3) is strictly convex, there exists a unique solution. Some approaches based on the following Euler–Lagrange equation:

$$\lambda(u-f) - \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \tag{1.4}$$

were proposed in [5–8]. In fact, these approaches used $\int_\Omega \sqrt{u_x^2+u_y^2+\epsilon}\,\mathrm{d}x$ to replace $\int_\Omega |\nabla u|\mathrm{d}x$ for $\epsilon > 0$ to avoid the non-differentiability of the total variation term in (1.3) at zero. Moreover, by means of the Legendre–Fenchel transformation, Chambolle [9] proposed using the semi-implicit gradient descent (or fixed point) algorithm to obtain the dual variable $\xi$; then the restored image can be efficiently obtained by using

$$u = f + \frac{1}{\lambda}\mathrm{div}\,\xi. \tag{1.5}$$

The semi-implicit gradient descent algorithm can effectively and quickly get the solution of the ROF model, so it has grown very popular [3,10,9,11].

However, the ROF model usually causes a staircase effect in the smooth regions although it can reduce oscillations and regularize the geometry of level sets without penalizing discontinuities. To decrease the staircase effect, some high-order derivative models have been recently introduced in image restoration [12,13,11]. In fact, high-order derivative models can be looked at as minimizing the total variation of the gradient, and so they can damp the oscillations much more quickly and require much stronger smoothness. One of the high-order derivative models proposed by Lysaker, Lundervold and Tai (called the LLT model) [12] is as follows:

$$\min_{u\in W^{2,1}(\Omega)\cap L^1(\Omega)} E_2(u) = \frac{\beta}{2}\int_\Omega (f-u)^2\mathrm{d}x + \int_\Omega |\nabla^2 u|\mathrm{d}x, \tag{1.6}$$

where $|\nabla^2 u| = \sqrt{u_{xx}^2+u_{xy}^2+u_{yx}^2+u_{yy}^2}$. Unfortunately, this model also tends to introduce some blurring in image edges as do other high-order derivative models.

In order to overcome the drawbacks of the ROF model and the LLT model, a two-step model has been proposed by Lysaker, Osher and Tai (called the LOT model) in [14]. They first used a smoothing vector $\mathbf{n} = (n_1, n_2)$ to fit the normal vector of the noisy image by solving the following problem:

$$\min_{|\mathbf{n}|=1} E_3(\mathbf{n}) = \frac{\gamma}{2}\int_\Omega \left(\mathbf{n} - \frac{\nabla f}{|\nabla f|}\right)^2 \mathrm{d}x + \int_\Omega |\nabla \mathbf{n}|\mathrm{d}x. \tag{1.7}$$

Let $\mathbf{n}^0$ be the solution of (1.7); the restored image $d$ is then reconstructed as the solution of the following minimization problem:

$$\begin{cases} \min\ E_4(d) = \displaystyle\int_\Omega (|\nabla d| - \mathbf{n}^0\cdot\nabla d)\mathrm{d}x, \\ \text{subject to } \|d-f\|_{L^2(\Omega)}^2 = \sigma^2, \end{cases} \tag{1.8}$$

where $\sigma$ is the standard deviation of the noise. Recently, some variants of the LOT method have also been studied, in [15–17]. But all of these models suffer from complicated implementations which are especially obvious in the first step (1.7). Motivated by the LOT model, we rewrite the Euler–Lagrange equation (1.4) as

$$u = f + \frac{1}{\lambda}\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{1.9}$$

Combining this with (1.5), it is easy to see that (1.9) implies that the dual variable $\xi$ in (1.5) can be formally looked at as the normal vector $\frac{\nabla u}{|\nabla u|}$ of the restored image. So we propose a new two-step model: we first obtain the dual variable $\xi$ and set it as the normal vector of the restored image; the restored image will be then restructured by fitting the normal vector. Furthermore, on the basis of the augmented Lagrangian strategy, we get the dual variable and the restored image by the projection gradient method and give some convergence analyses for the projection gradient method.

The organization of the paper is as follows. In Section 2, we introduce some results of the augmented Lagrangian strategy for solving the convex optimization problems. Following the augmented Lagrangian strategy, we propose the projection gradient method for solving our proposed two-step model and some convergence analyses are given in Section 3. The numerical implementations of the proposed model in the image denoising problems are shown in Section 4. In Section 5, some concluding remarks are given.