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Method of Lyapunov functions for differential equations with piecewise constant delay

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ABSTRACT

We address differential equations with piecewise constant argument of generalized type [5–8] and investigate their stability with the second Lyapunov method. Despite the fact that these equations include delay, stability conditions are merely given in terms of Lyapunov functions; that is, no functionals are used. Several examples, one of which considers the logistic equation, are discussed to illustrate the development of the theory. Some of the results were announced at the 14th International Congress on Computational and Applied Mathematics (ICCAM2009), Antalya, Turkey, in 2009.

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1. Introduction

Cooke, Wiener and their co-authors [1–4] introduced differential equations with piecewise constant argument, which play an important role in applications [5–13,1,2,14–19,3,4,20]. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with piecewise constant argument has been generalized in [5–7].

We should mention the following novelties of the present paper. The main and possibly a unique way of stability analysis for differential equations with piecewise constant argument has been the reduction to discrete equations [16,17,21–24,19,4]. Particularly, the problem of exploring stability with Lyapunov functions of continuous time has remained open. Moreover, the results of our paper have been developed through the concept of "total stability" [25,26], which is stability under persistent perturbations of the right-hand side of a differential equation, and they originate from a special theorem in [27]. Then, one can accept our approach as comparison of stability of equations with piecewise constant argument and ordinary differential equations. Finally, it deserves to emphasize that the direct method for differential equations with deviating argument necessarily utilizes functionals [28–30], but we use only Lyapunov functions to determine criteria of the stability, and this can be an advantage in applications.

2. The subject and method of analysis

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of natural numbers and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the n-dimensional real space by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the Euclidean norm in $\mathbb{R}^n$ by $\|.\|$.

Let us introduce a special notation:

$$\mathcal{K} = \{\psi : \psi \in C(\mathbb{R}^+, \mathbb{R}^+) \text{ is a strictly increasing function and } \psi(0) = 0\}.$$

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We fix a real-valued sequence $\theta_i, i \in \mathbb{N}$, such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots$ with $\theta_i \to \infty$ as $i \to \infty$, and shall consider the following equation

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{2.1}$$

where $x \in B(h), B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}, t \in \mathbb{R}^+$ and $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1}),\ i \in \mathbb{N}$, is an identification function.

We say that a continuous function $x(t)$ is a solution of Eq. (2.1) on $\mathbb{R}^+$ if it satisfies (2.1) on the intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{N}$ and the derivative $x'(t)$ exists everywhere with the possible exception of the points $\theta_i, i \in \mathbb{N}$, where one-sided derivatives exist.

In the rest of our paper, we assume that the following conditions hold:

(C1) $f(t, u, v) \in C(\mathbb{R}^+ \times B(h) \times B(h))$ is an $n \times 1$ real valued function;
(C2) $f(t, 0, 0) = 0$ for all $t \geq 0$;
(C3) f satisfies a Lipschitz condition with constants ℓ_1, ℓ_2:

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \ell_1 \|u_1 - u_2\| + \ell_2 \|v_1 - v_2\| \tag{2.2}$$

for all $t \in \mathbb{R}^+$ and $u_1, u_2, v_1, v_2 \in B(h)$;
(C4) there exists a constant $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta, i \in \mathbb{N}$;
(C5) $\theta[\ell_2 + \ell_1(1 + \ell_2\theta)e^{\ell_1\theta}] < 1$;
(C6) $\theta(\ell_1 + 2\ell_2)e^{\ell_1\theta} < 1$.

We give now some definitions and preliminary results which enable us to investigate stability of the trivial solution of (2.1).

Definition 2.1 ([7]). The zero solution of (2.1) is said to be

(i) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
(ii) uniformly stable if δ is independent of t_0.

Definition 2.2 ([7]). The zero solution of (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.

Next, we shall describe the method, which is in the base of our investigation. Let us rewrite the system (2.1) in the form

$$x'(t) = f(t, x(t), x(t)) + h(t, x(t), x(\beta(t))),$$

where $h(t, x(t), x(\beta(t))) = f(t, x(t), x(\beta(t))) - f(t, x(t), x(t))$. If the constant θ mentioned in (C4) is small, then we can consider $h(t, x(t), x(\beta(t)))$ as a small perturbation. That is to say, system (2.1) is a perturbed system for the following ordinary differential equation,

$$y'(t) = g(t, y(t)), \tag{2.3}$$

where $g(t, y(t)) = f(t, y(t), y(t))$.

Our intention is to consider systems (2.1) and (2.3) involved in the perturbation relation, and then extend these systems to the problem of stability based on the approach of Malkin [27].

Before applying the method, it is useful to consider a simple example. Let the following linear scalar equation with piecewise constant argument be given:

$$x'(t) = ax(t) + bx(\beta(t)) \tag{2.4}$$

where $\theta_i = ih,\ i \in \mathbb{N}$. The solution of (2.4) if $t \in [ih, (i+1)h)$ is given by [31,4]

$$x(t) = \left\{e^{a(t-ih)}\left(1 + \frac{b}{a}\right) - \frac{b}{a}\right\}\left\{e^{ah}\left(1 + \frac{b}{a}\right) - \frac{b}{a}\right\}^i x_0.$$

Then, one can easily see that the zero solution of (2.4) is asymptotically stable if and only if

$$-\frac{a(e^{ah} + 1)}{e^{ah} - 1} < b < -a. \tag{2.5}$$

On the other side, consider the following ordinary differential equation, which is associated with (2.4), and plays the role of (2.3),

$$y'(t) = ay(t) + by(t) = (a + b)y(t). \tag{2.6}$$

It is seen that the trivial solution of (2.6) is asymptotically stable if and only if

$$b < -a. \tag{2.7}$$

When the insertion of the greatest integer function is regarded as a "perturbation" of the linear equation (2.6), it is seen for (2.4) that the stability condition (2.5) is necessarily stricter than the one given by (2.7) for the corresponding "nonperturbed" equation (2.6). Moreover, it is seen that the condition (2.5) transforms to (2.7) as $h \to 0$.

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