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Extension of gamma, beta and hypergeometric functions

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ABSTRACT

The main object of this paper is to present generalizations of gamma, beta and hypergeometric functions. Some recurrence relations, transformation formulas, operation formulas and integral representations are obtained for these new generalizations.

1. Introduction

In recent years, some extensions of the well known special functions have been considered by several authors [1–7]. In 1994, Chaudhry and Zubair [1] have introduced the following extension of gamma function

$$\Gamma_p(x) := \int_0^\infty t^{x-1} \exp\left(-t - pt^{-1}\right) dt, \tag{1}$$

$$\mathrm{Re}(p) > 0.$$

In 1997, Chaudhry et al. [2] presented the following extension of Euler's beta function

$$B_p(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left[-\frac{p}{t(1-t)}\right] dt, \tag{2}$$

$$(\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0)$$

and they proved that this extension has connections with the Macdonald, error and Whittakers function. It is clearly seem that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x, y) = B(x, y)$. Afterwards, Chaudhry et al. [8] used $B_p(x, y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) as follows:

$$F_p(a, b; c; z) = \sum_{n=0}^\infty \frac{B_p(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!}$$

$$p \geq 0; \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$$

$$\phi_p(b; c; z) = \sum_{n=0}^\infty \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$p \geq 0; \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$$

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where $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \quad \text{and} \quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

and gave the Euler type integral representation

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t(1-t)}\right] dt$$
$$p > 0;\ p = 0 \quad \text{and} \quad |\arg(1-z)| < \pi < p;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

They called these functions extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. They have discussed the differentiation properties and Mellin transforms of $F_p(a,b;c;z)$ and obtained transformation formulas, recurrence relations, summation and asymptotic formulas for this function. Note that $F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$.

In this paper, we consider the following generalizations of gamma and Euler's beta functions

$$\Gamma_p^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt \tag{3}$$
$$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0,$$

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{4}$$
$$(\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0)$$

respectively. It is obvious by (1), (3) and (2), (4) that, $\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x)$, $\Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$, $B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y)$ and $B_0^{(\alpha,\beta)}(x,y) = B(x,y)$.

In Section 2, different integral representations and some properties of new generalized Euler's beta function are obtained. Additionally, relations of new generalized gamma and beta functions are discussed. In Section 3, we generalize the hypergeometric function and confluent hypergeometric function, using $B_p^{(\alpha,\beta)}(x,y)$ obtain the integral representations of this new generalized Gauss hypergeometric functions. Furthermore we discussed the differentiation properties, Mellin transforms, transformation formulas, recurrence relations, summation formulas for these new hypergeometric functions.

2. Some properties of gamma and beta functions

It is important and useful to obtain different integral representations of the new generalized beta function, for later use. Also it is useful to discuss the relationships between classical gamma functions and new generalizations. For $p = 0$, we have the following integral representation for $\Gamma_p^{(\alpha,\beta)}(x)$.

Theorem 2.1. *For the new generalized gamma function, we have*

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.$$

Proof. Using the integral representation of confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 u^{s-1}e^{-ut-\frac{pt}{u}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt\,du.$$

Now using a one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = ut$, $\mu = t$ in the above equality and considering that the Jacobian of the transformation is $J = \frac{1}{\mu}$, we get

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 v^{s-1}e^{-v-\frac{p\mu^2}{v}}dv\,\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \left[\int_0^\infty v^{s-1}e^{-v-\frac{p\mu^2}{v}}dv\right]\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$
$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.$$

Whence the result. □