Journal of Computational and Applied Mathematics 235 (2011) 4675–4686

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Approximating Cauchy-type singular integral by an automatic quadrature scheme

Z.K. Eskhuvatov [a,b,*], A. Ahmedov [b,c], N.M.A. Nik Long [a,b], N.J. Amalina [a]

[a] Department of Mathematics, Faculty of Science, University Putra of Malaysia (UPM), Malaysia
[b] Institute for Mathematical Research, UPM, Malaysia
[c] Department of Process and Food Engineering,Faculty of Engineering, UPM, Malaysia

ARTICLE INFO

*Article history:*
Received 13 December 2009
Received in revised form 25 July 2010

*MSC:*
65D32
45E05
65R20

*Keywords:*
An automatic quadrature scheme
Product integral
Singular integral
Clenshaw–Curtis rules
Chebyshev polynomials
Indefinite integral
Recurrence relation

ABSTRACT

An automatic quadrature scheme is developed for the approximate evaluation of the product-type indefinite integral

$$Q(f,x,y,c)=\int_x^y \rho(t)K(t,c)f(t)\,dt,\quad -1\le x,\ y\le 1,\ -1<c<1,$$

where $\rho(t)=1/\sqrt{1-t^2}$, $K(t,c)=1/(t-c)$ and $f(t)$ is assumed to be a smooth function. In constructing an automatic quadrature scheme, we consider two cases: (1) $-1<x<y<1$, and (2) $x=-1$, $y=1$. In both cases the density function $f(t)$ is replaced by the truncated Chebyshev polynomial $p_N(t)$ of the first kind of degree $N$. The approximation $p_N(t)$ yields an integration rule $Q_N(f,x,y,c)$ to the integral $Q(f,x,y,c)$. Interpolation conditions are imposed to determine the unknown coefficients of the Chebyshev polynomials $p_N(t)$. Convergence problem of the approximate method is discussed in the classes of function $C^{N+1,\alpha}[-1,1]$ and $L_p^w[-1,1]$. Numerically, it is found that when the singular point $c$ either lies in or outside the interval $(x,y)$ or comes closer to the end points of the interval $[-1,1]$, the proposed scheme gives a very good agreement with the exact solution. These results in the line of theoretical findings.



## 1. Introduction

Many methods have been developed for the numerical evaluation of a singular integral along the finite interval of the form

$$Q(f,c)=\int_b^a \rho(t)K(t,c)f(t)\,dt,\quad a<c<b, \tag{1}$$

where $\rho(t)$ is a weight function, $f(t)$ is a smooth function on the given interval $[a,b]$ and $K(t,c)$ is some singular or badly behaved kernel such as $\ln|t-c|$, $|t-c|^{\alpha}$, $\alpha>-1$, Cauchy kernel $\frac{1}{t-c}$, Hilbert kernel $\cot\left(\frac{t-c}{2}\right)$ or oscillatory kernel $e^{i\omega t}$, $|\omega|\gg 1$.

* Corresponding author at: Department of Mathematics, Faculty of Science, University Putra of Malaysia (UPM), Malaysia. Tel.: +60 3 89466855; fax: +60 3 89437958.
*E-mail address:* ezaini@science.upm.edu.my (Z.K. Eskhuvatov).


doi:10.1016/j.cam.2010.08.001

There exists a rich literature [1–14] on the numerical evaluation of the definite integral (1) for the different forms of the kernel $K(t,c)$ mentioned above. Unfortunately, for the numerical evaluation of the indefinite integrals of the form

$$Q(f,x,y,c)=\int_x^y \rho(t)K(t,c)f(t)\,dt,\quad -1\le x,\ y\le 1,\ -1<c<1, \tag{2}$$

a small literature is available [15–17].

Particularly, Rabinowitz [14] used the Jacobi polynomials $P_N(t)$ of degree $N$ to approximate the singular integral (SI) (1) when $\rho(t)=\rho^{(\alpha,\beta)}(t)=(1+t)^\alpha(1-t)^\beta,\alpha,\beta>-1$, kernel $K(t,c)=\frac{1}{t-c}$ and $[a,b]=[-1,1]$. They investigated the convergence of sequences of integration rules $Q_n(f,c)$ to $Q(f,c)$ for all $c\in(-1,1)$. Gauss–Jacobi rules and the interpolatory integration rules are used to construct the quadrature formulas (QFs) based on the zeros of $(1-t^2)P^{\alpha,\beta}_{n-2m}(t)$, $m=0,1$ for certain values of $\alpha$ and $\beta$. The results are then applied to the study of the convergence of Hunter's method for Cauchy principal value integrals. In 1990, Rabinowitz [13] consider product integral of the form (1) on the interval $J=[-1,1]$ with two types of kernels $\rho(t)K(t,c)=k(t),k(t)\in L_1(J)$ and $K(t,c)=\frac{1}{t-c}$. In both kernels convergence results are proved for product integration rules based on approximating splines. Pointwise and uniform convergence results are proved for sequences of Cauchy principal values of these approximating splines.

Diethelm [4] proposed a new QF $Q^{G3}_{n+1}[f;c]$, which is named the Gaussian quadrature formula of the third kind, defined by

$$Q[f,c]=\int_{-1}^1\frac{f(t)}{t-c}\,dt\cong Q^{G3}_{n+1}[f;c]=\int_{-1}^1\frac{\pi_{n-1}[f](t)-\pi_{n-1}[f](c)}{t-c}\,dt+f(c)\ln\frac{1-c}{1+c}, \tag{3}$$

where $\pi_{n-1}[f]$ is the interpolating polynomial of degree $n-1$ for the function $f$ at the zeroes of the $n$th Legendre polynomial with $P_n(t_i)=0$. In this QF he showed that this quadrature formula does not have the disadvantages of the other two well-known quadrature formulae based on the same set of nodes. In particular,he proved that sequence $Q^{G3}_{n+1}[f;c]$ converges to the true value of the integral $Q[f;c]$ uniformly for all $c\in(-1,1)$. In 1997, Diethelm [5] considers the Cauchy principal value integrals (1) with weight function $\rho(t)=w_{\alpha,\beta}(t)\psi(t)$, where $w_{\alpha,\beta}(t)=(1+t)^\alpha(1-t)^\beta$, $\alpha,\beta\ge-1/2$ is a Jacobi weight and $0<\psi\in DT$, where

$$DT:=\left\{\varphi\in C[-1,1]:\int_0^1\frac{\omega(\varphi,x)}{x}\,dx<\infty\right\}$$

is the class of Dini-type functions, and $\omega(\varphi,x)$ denotes the usual modulus of continuity of $\varphi$. A quadrature formula, so-called modified quadrature formula (MQF) is obtained by subtracting out the singularity and then apply a classical quadrature formula.He gave new bounds involving the total variation $\mathrm{Var} f^{(s)}$ and $L_p$ norms $\|f^{(s)}\|_p$ of some derivative of the integrand function and proved the uniform convergence of MQFs for all possible positions of the singular point.

In 2009, Eshkuvatov et al. [6] constructed QFs for SI (1) with weight function $\rho(t)=\frac{1}{\sqrt{1-t^2}}$ and kernel $K(t,c)=\frac{1}{t-c}$ on the interval $[-1,1]$, by combining the modified of discrete vortices method (MDV) and linear spline interpolation. They proved the uniform convergence of the QFs for all $c\in(-1,1)$ and improved the rate of convergence of MDV in the classes of functions $H^\alpha[-1,1]$ and $C^1[-1,1]$. In the same year Eshkuvatov et al. [7] have constructed QFs for SI (1) with $\rho(t)=1$ and $K(t,c)=\frac{1}{t-c}$ based on modified MDV, called MMDV, when the singular point $c$ lies in the middle of the subinterval $[t_j,t_{j+1}]$ or coincides with the knot points $c=t_j$ for fixed $j$. In this work they improved the rate of convergence of MDV in the classes of functions $H^\alpha[-1,1]$ and $C^1[-1,1]$.

Hasegawa and Torii [15] presented an automatic quadrature scheme (AQS) for the evaluation of indefinite integral of oscillatory function of the form $\int_0^x f(t)e^{i\omega t}\,dt$, $0\le x\le 1$, for a given function $f(t)$, which is assumed to be smooth. They combined an automatic quadrature scheme with Sidi's extrapolation method to make an effective quadrature scheme for oscillatory infinite integral $\int_0^\infty f(x)\cos\omega x\,dx$. In 1991, Hasegawa and Torii [9] proposed AQS for the product-type indefinite integral (2) with $\rho(t)=1$ and $K(t,c)$ are some singular or badly behaved functions such as $K(t,c)=|t-c|^\alpha,\alpha>-1$ or $K(t,c)=\ln|t-c|,-1\le c\le 1$ or $K(t,c)=e^{ict},|c|\gg 1$. In the derivation of AQS the function $f(t)$ is approximated by a truncated Chebyshev series $p_N(t)$ of degree $N$, whose coefficients are efficiently computed using the fast Fourier transform (FFT). The approximation $Q_N(f,x,y,c)$ to the integral $Q(f,x,y,c)$ is given by $Q(p_N,x,y,c)$. To enhance the efficiency of AQS, the degree $N$ is increased more slowly than doubling. The evaluations of $Q_N(f,x,y,c)=Q(p_N,x,y,c)$ for a set of $(x,y,c)$ can be efficiently made by using recurrence relations for the singular kernels $K(t,c)$ above.

In 2007, Hasegawa and Suguira [17] proposed AQS for approximating the indefinite integral having algebraic–logarithmic singularities

$$Q(f,x,y,c)=\int_x^y f(t)|t-c|^\alpha\log|t-c|\,dt,\quad -1\le x,y,c\le 1,\ \alpha>-1, \tag{4}$$

within a finite range $[-1,1]$ for some smooth function $f(t)$. They expand the given indefinite integral in terms of Chebyshev polynomials by using auxiliary algebraic–logarithmic functions. The scheme approximates the indefinite integral $Q(f,x,y,c)$ uniformly, independently of the value $c$ as well $x$ and $y$. This fact enables them to evaluate the integral transform $Q(f,x,y,c)$ with varied values of $x$, $y$ and $c$ efficiently. In the same paper they proposed an open problem for the uniform convergence of AQS for SI (5).