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Solving optimal control problems for the unsteady Burgers equation in COMSOL Multiphysics

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ABSTRACT

The optimal control of unsteady Burgers equation without constraints and with control constraints are solved using the high-level modelling and simulation package COMSOL Multiphysics. Using the first-order optimality conditions, projection and semi-smooth Newton methods are applied for solving the optimality system. The optimality system is solved numerically using the classical iterative approach by integrating the state equation forward in time and the adjoint equation backward in time using the gradient method and considering the optimality system in the space–time cylinder as an elliptic equation and solving it adaptively. The equivalence of the optimality system to the elliptic partial differential equation (PDE) is shown by transforming the Burgers equation by the Cole–Hopf transformation to a linear diffusion type equation. Numerical results obtained with adaptive and nonadaptive elliptic solvers of COMSOL Multiphysics are presented both for the unconstrained and the control constrained case.

1. Introduction

The Burgers equation plays an important role in fluid dynamics as a first approximation to complex diffusion–convection phenomena. It was used as a simplified model for turbulence and in shock waves. Analysis and numerical approximation of optimal control problems for the Burgers equation are important for the development of numerical methods for optimal control of more complicated models in fluid dynamics like Navier–Stokes equations.

Recently, several papers appeared dealing with the optimal control of the Burgers equation. A detailed analysis of distributed and boundary control of stationary and unsteady Burgers equation and the approximation of the optimality system with augmented Lagrangian SQP (sequential quadratic programming) method are given in [1]. In [2], the SQP, primal–dual active set and semi-smooth Newton methods are compared for distributed control problems related with the stationary Burgers equation with pointwise control constraints. Distributed control problems for the unsteady Burgers equation with and without control constraints are investigated numerically using SQP methods in [3–5]. Different time integration methods like the implicit Euler and Crank–Nicolson methods were considered for solving the adjoint equations arising from the optimal control of the unsteady Burgers equation in [6]. In contrast to linear parabolic control problems, the optimal control problem for the Burgers equation is a non-convex problem with multiple local minima due to the nonlinearity of the differential equation. Numerical methods can only compute minima close to the starting points [4].

Parabolic optimal control problems with and without constraints were solved using COMSOL Multiphysics [7–9]. In this paper, we present numerical results for distributed optimal control of the unsteady Burgers equation without and with

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control constraints. We follow the function based "first optimize then discretize" strategy which allows us to apply different optimization techniques for solving the optimality conditions.

For parabolic optimal control problems, the optimality system contains a forward and a backward-in-time equation coupled by an algebraic equation. The optimality system can be solved by the gradient method integrating the state equation forward in time and the adjoint equation backward in time iteratively. Another approach which appeared recently is to consider the time as an additional space dimension and solve the elliptic PDE that contains the whole optimality systems by COMSOL Multiphysics as in [7–9].

The paper is organized as follows. In Section 2, the distributed optimal control problem for the unconstrained Burgers equation is stated and the optimality system is obtained. In Section 3, we give the gradient method for solving the optimality system and present numerical results obtained by COMSOL Multiphysics. The so-called one-shot approach by transforming the whole optimality system to an elliptic PDE is given in Section 3. We make use of the Cole–Hopf transformation to obtain the elliptic equation for the linearized Burgers equation. Numerical results with adaptive and nonadaptive solvers of COMSOL Multiphysics are given for different mesh sizes. In Section 4, we give the optimality system for the Burgers equation with pointwise control constraints. For solving the optimality system, the projection method with semi-smooth Newton method was used. The paper ends with some conclusions in Section 5.

2. Optimality system for the Burgers equations without inequality constraints

We summarize first the existence and uniqueness of solutions of the unsteady Burgers equation following [4,10]. Given $\Omega = (0, 1)$ and $T > 0$, we define $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ be Hilbert spaces. We make use of the following Hilbert space:

$$W(0, T) = \{\varphi \in L^2(0, T; V); \varphi_t \in L^2(0, T; V^*)\},$$

where V^* denotes the dual space of V. The inner product in the Hilbert space V is given with the natural inner product in H as

$$(\varphi, \psi)_V = (\varphi', \psi')_H, \quad \text{for } \varphi, \psi \in V.$$

The expression $\varphi(t)$ stands for $\varphi(t, \cdot)$, considered as function in Ω only when t is fixed.

We consider the unsteady viscous Burgers equation

$$y_t + yy_x - \nu y_{xx} = f + bu \quad \text{in } Q \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$y(t, 0) = 0 \quad \text{on } \Sigma$$

and with the initial condition

$$y(0) = y_0 \quad \text{in } \Omega$$

where $f \in L^2(Q)$ is a fixed forcing term, $\nu = \frac{1}{Re} > 0$ denotes the viscosity parameter and *Re* is the Reynolds number. The location and intensity of the controls $u \in L^2(Q)$ are expressed by the function $b \in L^\infty(Q)$. For example b might be chosen as

$$bu = \begin{cases} u & \text{in } \tilde{\Omega} \\ 0 & \text{in } \Omega \setminus \tilde{\Omega} \end{cases}$$

where $\tilde{\Omega}$ is the set of active controls [2,4,1].

For the unsteady Burgers equation (1) with the corresponding initial and boundary conditions there exists a weak solution $y \in W(0, T)$ satisfying

$$\langle y_t(t), \varphi\rangle_{V^*,V} + \nu(y_t(t), \varphi)_V + (y(t)y_x(t), \varphi)_H = ((f + bu)(t), \varphi)_H$$

for all $\varphi \in V$, and $t \in [0, T]$, and $(y(0), \chi)_H = (y_0, \chi)$ for all $\chi \in H$ (see [4]).

The distributed control problem for the Burgers equation without inequality constraints and with homogeneous Dirichlet boundary conditions *(P)* can be stated as follows [1]:

$$\begin{aligned} &\min \quad J(y, u) = \frac{1}{2}\|y - y_d\|_Q^2 + \frac{\alpha}{2}\|u\|_Q^2 \\ &\text{s.t.} \quad y_t + yy_x - \nu y_{xx} = f + bu \quad \text{in } Q, \\ &y = 0 \quad \text{on } \Sigma, \\ &y(0) = y_0 \quad \text{in } \Omega, \end{aligned} \tag{2}$$

with the regularization parameter $\alpha > 0$. Here, y and u denote the state and control variables, y_d is the desired state.

In order to show the existence of the optimal solutions, the operator $e : X \to Y$ (see pp. 130 [10]) was introduced by

$$e(y, u) = (e_1(y, u), e_2(y, u)) = (y_t - \nu y_{xx} + yy_x - f - bu, y(0) - y_0),$$