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Contractivity of domain decomposition splitting methods for nonlinear parabolic problems☆

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ABSTRACT

This work deals with the efficient numerical solution of nonlinear parabolic problems posed on a two-dimensional domain Ω. We consider a suitable decomposition of domain Ω and we construct a subordinate smooth partition of unity that we use to rewrite the original equation. Then, the combination of standard spatial discretizations with certain splitting time integrators gives rise to unconditionally contractive schemes. The efficiency of the resulting algorithms stems from the fact that the calculations required at each internal stage can be performed in parallel.

1. Introduction

The aim of this paper is to design robust and efficient methods for the numerical solution of nonlinear multidimensional parabolic problems governed by an equation of this type:

$$\frac{\partial u}{\partial t} = f(t, \underline{x}, u, \nabla \cdot (K(t, \underline{x})\nabla u)), \quad (t, \underline{x}) \in (0, T] \times \Omega. \tag{1}$$

Within this framework, the drawbacks of classical discretization methods are well known. On the one hand, if we choose an explicit method, very small time steps must be taken in order to ensure a stable integration, especially if fine meshes are considered to discretize in space. On the other hand, if we use suitable implicit methods, arbitrarily large time steps may be considered without loosing stability; nevertheless, computationally expensive iterative methods are required to solve the large nonlinear systems arising at each time step.

As an efficient alternative, Verwer proposed in [1] two splitting methods that were shown to be unconditionally contractive for a large class of nonlinear parabolic problems. In particular, the author dealt with classical alternating direction splittings that gave rise to locally one-dimensional methods. The main advantage of these algorithms is that the resulting nonlinear systems can be solved very cheaply using Newton iteration since all the Jacobian matrices can be permuted to a banded form (one-dimensional structure). Moreover, these systems are easily uncoupled into much smaller linear subsystems that can be solved in parallel.

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Taking into account these advances, the idea of this work is to combine the same time integrators with another splitting technique, which is related to a suitable decomposition of the spatial domain. This type of splittings, in combination with fractional step methods, has been analyzed for the numerical solution of linear parabolic problems (cf. [2–4]). Unfortunately, the stability results proven for linear problems cannot be extended to the nonlinear case. Nevertheless, the fact that our schemes can be included in the general framework studied by J. Verwer will allow us to derive contractivity results for some of them also in the nonlinear case.

Regarding the good properties of the methods proposed in this paper, it is important to notice that each implicit internal stage can be efficiently computed using Newton iteration since the linear systems to solve at each iteration are, in fact, a collection of much smaller linear systems that can be solved in parallel. Moreover, in the case of considering semilinear parabolic problems, we show that our proposal is also suitable for equations where mixed derivative terms are present, contrary to what happens with classical alternating direction implicit (ADI) techniques.

The structure of the paper is as follows. Section 2 compiles some definitions and theoretical results that will be used throughout the paper. The next section studies the contractivity in the maximum norm of our proposal for a large class of nonlinear parabolic problems. Section 4 contains contractivity results in the discrete two-norm for the case of semilinear parabolic problems with mixed derivative terms. Finally, the last section contains two numerical examples that illustrate the convergence of the proposed algorithms.

2. Preliminaries

Let us consider the following ordinary differential system

$$\dot{U} = F(t, U), \quad t \in (0, T], \tag{2}$$

together with a given initial condition, where $F : [0, T] \times \mathbb{R}^s \to \mathbb{R}^s$ is continuous in t and continuously differentiable in U on $[0, T] \times \mathbb{R}^s$. Let us denote with $\|\cdot\|$ a norm on $\mathbb{R}^s$ as well as its subordinate matrix norm on $\mathbb{R}^{s\times s}$. Given an $s \times s$ matrix A, its logarithmic norm with respect to $\|\cdot\|$ is defined as follows

$$\mu[A] = \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}.$$

Therefore, the actual value of $\mu[A]$ depends on the norm on which $\mu[A]$ is based. In this work, we shall deal with the logarithmic norms associated to $\|\cdot\|_2$ and $\|\cdot\|_\infty$, which can be computed as (cf. [5])

$$\mu_2[A] = \text{ maximal eigenvalue of } \frac{1}{2}(A + A^T), \tag{3}$$

$$\mu_\infty[A] = \max_{i=1,2,\ldots,s} \left(a_{ii} + \sum_{j\neq i} |a_{ij}| \right), \tag{4}$$

respectively, where $A \equiv (a_{ij}) \in \mathbb{R}^{s\times s}$. However, the following property holds true for any $s \times s$ matrix A and any associate norm: $\mu[A] \geqslant \alpha[A]$, where $\alpha[A]$ denotes the maximal real part of the eigenvalues of A. In this framework, a norm is called logarithmically optimal with respect to a matrix A if $\mu[A] = \alpha[A]$ (cf. [6] and references therein).

The concept of logarithmic norms, that was introduced in the fifties independently by G. Dahlquist and S.M. Lozinskij, is of great use in the perturbation analysis of nonlinear differential equations, as is shown in the following

Proposition 2.1 (*Dahlquist [5]*). *Let $\|\cdot\|$ be a given norm. Let $\nu : [0, T] \to \mathbb{R}$ be a piecewise continuous function satisfying $\mu\left[\frac{\partial F(t,\xi)}{\partial U}\right] \leqslant \nu(t)$ for every $(t, \xi) \in [0, T] \times \mathbb{R}^s$. Then, for any two solutions U and $\tilde{U}$ of (2),*

$$\|\tilde{U}(t_2) - U(t_2)\| \leqslant e^{\int_{t_1}^{t_2} \nu(r)\mathrm{d}r} \|\tilde{U}(t_1) - U(t_1)\|, \quad \forall\, 0 \leqslant t_1 \leqslant t_2 \leqslant T.$$

Therefore, if $\mu\left[\frac{\partial F(t,\xi)}{\partial U}\right] \leqslant 0$ for every $(t, \xi) \in [0, T] \times \mathbb{R}^s$, we can take $\nu(t) \equiv 0$ and $\|\tilde{U}(t_2) - U(t_2)\| \leqslant \|\tilde{U}(t_1) - U(t_1)\|\ \forall\, 0 \leqslant t_1 \leqslant t_2 \leqslant T$. Differential systems with this property are called dissipative.

The freedom of choosing different matrix norms for the logarithmic norm can be exploited for two purposes. On the one hand, such a norm should be natural for the problem under study and, on the other hand, it should be selected in such a way that the upper bound $\nu(t)$ is as small as possible.

Next, let us introduce the following definition for function F

Definition 2.1. Let $F(t, U)$, where $F : [0, T] \times \mathbb{R}^s \to \mathbb{R}^s$, be continuous in t and continuously differentiable in U on $[0, T] \times \mathbb{R}^s$. Then, F is called dissipative with respect to a given norm $\|\cdot\|$ if $\mu\left[\frac{\partial F(t,\xi)}{\partial U}\right] \leqslant 0\ \forall\, (t, \xi) \in [0, T] \times \mathbb{R}^s$.