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Stieltjes transforms defined by C_0-semigroups

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In this paper we use the resolvent semigroup associated to a C_0-semigroup to introduce the vector-valued Stieltjes transform defined by a C_0-semigroup. We give new results which extend known ones in the case of a scalar generalized Stieltjes transform. We work with the vector-valued Weyl fractional calculus to present a deep connection between both concepts.

0. Introduction

Integral transforms are useful techniques to study integral and differential equations. One of the most famous is the Laplace transform, but other ones like the Fourier, Mellin or Hankel transforms are used in different problems.

The Stieltjes transform of a function $f : [0, \infty) \to \mathbf{C}$ is defined by the integral expression

$$\mathcal{S}f(t) := \int_0^\infty \frac{f(s)}{s+t}\mathrm{d}s, \quad t > 0.$$

Its connection with the Laplace transform and some results about inversion are given, for example in [1]. The generalized Stieltjes transform for $\alpha > 0$ defined by

$$\mathcal{S}_\alpha f(t) := \int_0^\infty \frac{f(s)}{(s+t)^\alpha}\mathrm{d}s, \quad t > 0,$$

is a natural extension studied in [2]. A well-written survey about the Stieltjes transform of generalized functions with different applications to differential equations may be found in [3].

The vector-valued Stieljes transform has been introduced in [4] where some results about the inversion are shown. Note that our starting point is different. Given a uniformly bounded C_0-semigroup of linear and bounded operators, this short note is mainly dedicated to extend the generalized Stieltjes transform.

A C_0-semigroup of linear and bounded operators $(T(t))_{t\geq 0}$ is a natural setting to extend the Stieltjes transform. Many integro-differential operators generate C_0-semigroups and the relationship between Cauchy problems and C_0-semigroups is well-known [5].

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In the first section we recall some basic properties about the fractional semigroup and scalar Stieltjes and Laplace transforms. Algebraic structure for the usual convolution product in the semigroup $[0,\infty)$ is considered to use it in next sections.

In the second section, we present a short introduction of C_0-semigroups and resolvent semigroups. The result Theorem 2.1 is original and lets us give solutions of abstract integral equations and show new results in the fourth section.

In the third section we present some facts about the vector-valued Weyl fractional calculus, and its connection with resolvent semigroups, see for example Proposition 3.1.

In the last section, we present main results of the paper. We introduce the Stieltjes transform associated to a C_0-semigroup, we give some basic results and we give a deep connection between Weyl fractional calculus and Stieltjes transform, see Theorem 4.5.

Notation. $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ are the set of natural, real and complex numbers; $\Re z$ is the real part of a complex number z; $\mathbf{C}^+ := \{z \in \mathbf{C} \mid \Re z > 0\}$.

Γ is the Euler function, and χ_E is the characteristic function on the subset E. Let X be a Banach space. We denote by $\mathcal{S}_+(X)$ the Schwartz class on $[0,\infty)$ i.e., functions $f : [0,\infty) \to X$, which are infinitely differentiable and verify

$$\sup_{t\geq 0} \left\| t^m \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) \right\| < \infty$$

for any $m, n \in \mathbf{N} \cup \{0\}$. We write by $\mathcal{S}_+ = \mathcal{S}_+(\mathbf{C})$.

1. Fractional semigroup and integral transforms in $L^1(\mathbf{R}^+)$

We denote by $L^1(\mathbf{R}^+)$ the usual Lebesgue space of (class of) measurable functions f such that

$$\|f\|_1 := \int_0^\infty |f(t)| < \infty.$$

It is well know that $(L^1(\mathbf{R}^+), \|\ \ \|_1)$ is a Banach algebra with the convolution product $*$ given by

$$f * g(t) = \int_0^t f(t-s)g(s)\mathrm{d}s, \quad f, g \in L^1(\mathbf{R}^+).$$

The Laplace transform $\mathcal{L} : L^1(\mathbf{R}^+) \to H^\infty(\overline{\mathbf{C}^+})$, given by

$$\mathcal{L}f(z) := \int_0^\infty f(t)\mathrm{e}^{-zt}\mathrm{d}t, \quad z \in \overline{\mathbf{C}^+},$$

is a linear and bounded algebra homomorphism, where $H^\infty(\overline{\mathbf{C}^+})$ is the algebra of analytic and bounded functions in $\overline{\mathbf{C}^+}$ with a pointwise product.

Theorem 1.1 ([6, Theorem 2.6]). *We consider for $z \in \mathbf{C}^+$ the function I^z given by*

$$I^z(t) := \frac{t^{z-1}}{\Gamma(z)}\mathrm{e}^{-t}, \quad z \in \mathbf{C}^+.$$

Then it holds that

(i) $I^z * I^w = I^{z+w}$, *for* $z, w \in \mathbf{C}^+$;
(ii) *The set* $I^z * L^1(\mathbf{R}^+)$ *is dense in* $L^1(\mathbf{R}^+)$;
(iii) $\|I^z\|_1 = \frac{\Gamma(\Re z)}{|\Gamma(z)|}$;
(iv) $\mathcal{L}I^z(w) = (1+w)^{-z}$, *for* $z, w \in \mathbf{C}^+$.

The family $(I^z)_{\Re z>0}$ is called the fractional semigroup in $L^1(\mathbf{R}^+)$.

Close to the fractional semigroup, we may define functions $(e_{\varepsilon,\alpha})_{\varepsilon,\alpha>0}$ by

$$e_{\varepsilon,\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-\varepsilon t}, \quad t > 0.$$

Note that $(e_{\varepsilon,\alpha})_{\varepsilon,\alpha>0} \subset L^1(\mathbf{R}^+)$, $e_{\varepsilon,\alpha}(t) = \frac{1}{\varepsilon^{\alpha-1}}I^\alpha(\varepsilon t)$ and $\|e_{\varepsilon,\alpha}\|_1 = \frac{1}{\varepsilon^\alpha}$ for $\varepsilon, \alpha > 0$.