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Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind ☆

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ABSTRACT

In this paper we propose new numerical methods for linear Fredholm integral equations of the second kind with weakly singular kernels. The methods are developed by means of the Sinc approximation with smoothing transformations, which is an effective technique against the singularities of the equations. Numerical examples show that the methods achieve exponential convergence, and in this sense the methods improve conventional results where only polynomial convergence have been reported so far.

1. Introduction

The purpose of this paper is to develop high order numerical methods for Fredholm integral equations of the form

$$\lambda u(t) - \int_a^b |t-s|^{p-1} k(t,s) u(s) \mathrm{d}s = g(t), \quad a \leq t \leq b, \tag{1.1}$$

where $\lambda \neq 0,\ 0 < p < 1$, k and g are given functions, and u is the solution to be determined. Equations of this form often arise in practical applications such as Dirichlet problems, mathematical problems of radiative equilibrium and radiative heat transfer problems [1–3].

The construction of high order methods for the equations is, however, not an easy task because of the singularity in the "weakly singular" kernel $|t-s|^{p-1}k(t,s)$ (note that $p < 1$); in fact, in this case the solution u is generally not differentiable at the endpoints (i.e. $t = a$ and $t = b$) [1,4–6], and due to this, to the best of the authors' knowledge the best convergence rate ever achieved remains only at polynomial order. For example, if we set uniform meshes with $n + 1$ grid points and apply the spline methods of order m, then the convergence rate is only $\mathrm{O}(n^{-2p})$ at most [2,7], and it cannot be improved by increasing m. One way of remedying this is to introduce graded meshes [2,7,8]. Then the rate is improved to $\mathrm{O}(n^{-m})$ [8,9] which now depends on m, but still at polynomial order. Furthermore, as pointed in [10], this idea contains several substantial drawbacks such as that the implementation is complicated compared to the case of uniform meshes, and that the system of linear equations generated in this way becomes rapidly ill-conditioned as m increases. To counter these issues,

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Monegato–Scuderi [10] have proposed to introduce a smoothing transformation, instead of graded meshes, with which the solution can be made arbitrarily smooth. Then with the standard spline method on uniform meshes the better rate $\mathrm{O}(n^{-m})$ can be obtained without the drawbacks above (for the same reason, the concept of a smoothing transformation has recently been used by several authors [11–13]). Other methods for Eq. (1.1) include [14,15], whose convergence rates are all of polynomial.

On the other hand, a method with *exponential* order convergence rate has been developed in [16] for Volterra integral equations of the form

$$u(t) - \int_a^t (t-s)^{p-1} k(t,s) u(s) \mathrm{d}s = g(t), \quad a \leq t \leq b,$$

where the kernel is also assumed to be weakly singular, and the solution u is generally not differentiable at $t = a$ (cf. [17]). The key here is to utilize not only the concept of a smoothing transformation described above but this time also the so-called Sinc approximation; this is motivated by the fact that the combination is generally an effective tool for functions with derivative singularity at endpoints (cf. [18]). Riley then confirmed numerically that his method in fact achieves exponential convergence $\mathrm{O}(\exp(-c_1\sqrt{n}))$ despite the singularity. Furthermore, it can be examined numerically that in his method the system of linear equations is very well-conditioned.

With these backgrounds, we propose two new numerical methods for Eq. (1.1). The first method is given by simply extending Riley's idea to the Fredholm case. It is then shown by numerical experiments that the new method enjoys the same convergence rate $\mathrm{O}(\exp(-c_1\sqrt{n}))$ as in the Volterra case. The second method is derived by replacing the smoothing transformation employed in the first method, the standard *tanh transformation*, with the so-called *double exponential transformation*. This modification is motivated by an observation that in various cases [19,20] such replacement drastically improves the order of convergence. In fact, it turns out that the modification works well also in our case, and numerical experiments suggest that the convergence rate is improved to $\mathrm{O}(\exp(-c_2 n/\log n))$. In both of the new methods the linear equations are very well-conditioned. Finally, we also give a way of estimating a tuning parameter d which is the most essential parameter in the methods and substantially affects their performance. We like to emphasize that this point has been left unanswered in [16].

This paper is organized as follows. In Section 2, we state basic theorems of the Sinc methods, which are referred to in the subsequent sections. In Section 3, two numerical methods are derived by means of the Sinc approximation. In Section 4, we analyze the regularity of the solution u of Eq. (1.1). In Section 5, we give error bounds of the proposed methods. In Section 6 we show numerical results. Finally in Section 7 we conclude this paper.

2. Basic definitions and theorems of Sinc methods

2.1. Sinc approximation

The original Sinc approximation is expressed as

$$f(x) \approx \sum_{j=-N}^{N} f(jh) S(j,h)(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where the basis function $S(j,h)(x)$ (the so-called *Sinc function*) is defined by

$$S(j,h)(x) = \frac{\sin \pi(x/h - j)}{\pi(x/h - j)},$$

and h is a step size appropriately chosen depending on a given positive integer N. Note that the approximation formula (2.1) is defined on $x \in \mathbb{R}$, whereas the target Eq. (1.1) is defined on the finite interval (a, b). In order to relate these two intervals, $\mathbb{R}$ and (a, b), the *tanh transformation* (and its inverse)

$$t = \phi^{\mathrm{SE}}_{a,b}(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$

$$x = \{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(t) = \log\left(\frac{t-a}{b-t}\right)$$

can be introduced [18]. Throughout this paper, we call this the *single exponential (SE) transformation*, and the combination of (2.1) and the SE transformation the *SE-Sinc approximation*.

In order that the formula (2.1) on $\mathbb{R}$ works accurately, a function to be approximated should be analytic on a strip domain, $\mathscr{D}_d = \{z \in \mathbb{C} : |\mathrm{Im} z| < d\}$ for some $d > 0$, and also should be bounded in some sense. When incorporated with the SE transformation, the conditions should be considered on the translated domain

$$\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \right\}.$$

In order to clarify the conditions more precisely, it is convenient to introduce the following function space.