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On block-circulant preconditioners for high-order compact approximations of convection–diffusion problems

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ABSTRACT

We study some properties of block-circulant preconditioners for high-order compact approximations of convection–diffusion problems. For two-dimensional problems, the approximation gives rise to a nine-point discretisation matrix and in three dimensions, we obtain a nineteen-point matrix. We derive analytical expressions for the eigenvalues of the block-circulant preconditioner and this allows us to establish the invertibility of the preconditioner in both two and three dimensions. The eigenspectra of the preconditioned matrix in the two-dimensional case is described for different test cases. Our numerical results indicate that the block-circulant preconditioning leads to significant reduction in iteration counts and comparisons between the high-order compact and upwind discretisations are carried out. For the unpreconditioned systems, we observe fewer iteration counts for the HOC discretisation but for the preconditioned systems, we find similar iteration counts for both finite difference approximations of constant-coefficient two-dimensional convection–diffusion problems.

1. Introduction

We consider block-circulant preconditioners for linear systems arising from high-order compact (HOC) discretisations of the steady-state convection–diffusion problem

$$-\epsilon\Delta u(\mathbf{x}) + \mathbf{w}(\mathbf{x})\cdot\nabla u(\mathbf{x}) = f(\mathbf{x}), \tag{1}$$

in a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Dirichlet boundary conditions $u(\mathbf{x}) = g(\mathbf{x})$ on $\partial\Omega$. In (1), $\epsilon > 0$ is the diffusivity parameter, $\mathbf{w}$ is the convective velocity field and $f(\mathbf{x})$ is the source term. In the convection-dominated case ($\|\mathbf{w}\| \gg \epsilon$), the solution has steep gradients or exhibits interior layers in some parts of the domain Ω. When these are due to the Dirichlet boundary conditions on the outflow boundaries, the numerical solutions arising from standard finite difference (central difference scheme) or finite element (Galerkin scheme) discretisations exhibit non-physical oscillations. In order to produce stable discrete solutions, a modification of the Galerkin scheme is the SUPG (streamline upwind Petrov–Galerkin) scheme of [1] which stabilizes the discrete variational formulation through the addition of consistent terms proportional to the residual of the discrete solution on each element.

In the context of finite difference discretisations, one way to avoid the phenomenon of instability is to use an upwind discretisation for the first-order derivative. However, the upwind scheme is only first-order accurate and therefore the solutions of large linear systems are required to obtain sufficiently accurate numerical solutions.

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A suitable alternative to the upwind scheme for problems in two dimensions is the high-order compact discretisation introduced in [2]. This nine-point scheme has been shown to produce numerical solutions with high accuracy when the cell-Reynolds number is not too large. Similar schemes of $\mathcal{O}(h^4)$ accuracy for Navier–Stokes equations have been developed in [3,4] and these schemes have been shown to yield highly accurate numerical solutions.

For three-dimensional problems, a different mechanism for deriving a nineteen-point high-order compact scheme was described in [5]. An interesting property for both the nine and nineteen-point schemes are that they have been observed to yield non-oscillatory discrete solutions for various test problems even in the convection-dominated case. For grid-aligned flows, this non-oscillatory property has been theoretically established in [6] for the two-dimensional case and in [7] for a three-dimensional problem.

The study of iterative solution methods for HOC linear systems arising from two-dimensional problems has revealed some interesting properties of the nine-point scheme. The unconditional stability of this scheme with the Jacobi relaxation scheme has resulted in the development of efficient multigrid techniques [8]. An analysis of the stability of incomplete factorisations for a problem with grid-aligned constant flow carried out in [9] has shown that contrary to the case of central difference approximations where the performance of ILU-preconditioned iterations are adversely affected by unstable triangular solves in the convection-dominated case, such problems are not observed for the high-order scheme.

The performance of preconditioners based on incomplete factorisations and sparse approximate inverses for solving linear systems arising from HOC approximations of two-dimensional problems has been studied in [10]. In this present work, we consider block-circulant preconditioning of the HOC linear systems. The invertibility of the preconditioners are established and we then give an experimental study of the eigenspectra of the preconditioned matrix in the two-dimensional case. A brief outline is described next.

In Section 3, we recall some properties of circulant matrices and in Section 2, we give the discretisation matrices for grid-aligned flow problems in two and three dimensions and we give analytical expressions for their eigenvalues. In Section 4 we analyze a block-circulant preconditioner for the two-dimensional case and we prove that the preconditioner is invertible. In Section 5, a similar analysis is carried out for the three-dimensional case. Numerical results on the performance of the block-circulant preconditioner for the iterative solution of HOC linear systems are given in Section 6.

In the rest of the paper, e_j denotes the jth canonical basis vector, I denotes the identity matrix, $C \otimes L$ is the Kronecker product of matrices C and L and depending on the dimension d, the discretisation matrix of order $N = n^d$ will be denoted by A.

2. High-order compact schemes

We consider problems when the flow is constant and aligned with the grid on $\Omega = (0, 1)^d$. We thus assume that in (1), $\epsilon = 1$ and that the velocity field is given by $\mathbf{w} = (\tau, 0)$ for $d = 2$ and $\mathbf{w} = (\tau, 0, 0)$ when $d = 3$.

Using a columnwise ordering of the grid unknowns on a mesh with spacing $h = 1/(n + 1)$ in each direction, and letting $\gamma = \tau h/2$ denote the cell-Reynolds number, the HOC system matrix A when $d = 2$ has the structure

$$A = \text{blocktridiag}\,[L,\ K,\ M]\,, \tag{2}$$

where L, K and M are the $n \times n$ tridiagonal matrices given by

$$\begin{aligned} L &= \text{tridiag}\left[-(1+\gamma),\ -(4+4\gamma+2\gamma^2),\ -(1+\gamma)\right], \\ K &= \text{tridiag}\left[-4,\ 20+4\gamma^2,\ -4\right], \\ M &= \text{tridiag}\left[-(1-\gamma),\ -(4-4\gamma+2\gamma^2),\ -(1-\gamma)\right]. \end{aligned}$$

Since L, K and M are symmetric tridiagonal matrices, they all have the same eigenvector matrix $V = \left(v_{ij}\right)_{1\le i,\, j\le n}$ with $v_{ij} = \sin ij\pi h$. This result leads to the following analytical expressions for the eigenvalues $\left(\theta_{jk}\right)_{1\le j,k\le n}$ of the matrix A in (2) [6]

$$\theta_{jk} = \lambda_j^{(K)} + 2\sqrt{\lambda_j^{(L)}\lambda_j^{(M)}}\cos k\pi h,$$

where the eigenvalues $\lambda_j^{(L)}$, $\lambda_j^{(K)}$ and $\lambda_j^{(M)}$ of the matrices L, K and M for $j = 1, 2, \ldots, n$ are respectively given by

$$\begin{aligned} \lambda_j^{(L)} &= -2\left(1+(\gamma+1)^2\right) - 2(1+\gamma)\cos j\pi h, \\ \lambda_j^{(K)} &= 20 + 4\gamma^2 - 8\cos j\pi h, \\ \lambda_j^{(M)} &= -2\left(1+(\gamma-1)^2\right) - 2(1-\gamma)\cos j\pi h. \end{aligned} \tag{3}$$

Using (3) it then follows that the n^2 eigenvalues of the system matrix A arising from the grid-aligned flow problem can be written in the form

$$\theta_{jk} = 20 + 4\gamma^2 - 8\cos j\pi h + 4\cos k\pi h\sqrt{4+\gamma^4+\left(4-2\gamma^2\right)\cos j\pi h + \left(1-\gamma^2\right)\cos^2 j\pi h}, \quad 1 \le j, k \le n. \tag{4}$$

In [6], it is proved that the eigenvalues θ_{jk} are positive for all values of the cell-Reynolds number γ.