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A class of algebraic–trigonometric blended splines

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ABSTRACT

This paper presents a new kind of algebraic–trigonometric blended spline curve, called *xyB* curves, generated over the space $\{1, t, \sin t, \cos t, \sin^2 t, \sin^3 t, \cos^3 t\}$. The new curves not only inherit most properties of usual cubic B-spline curves in polynomial space, but also enjoy some other advantageous properties for modeling. For given control points, the shape of the new curves can be adjusted by using the parameters x and y. When the control points and the parameters are chosen appropriately, the new curves can represent some conics and transcendental curves. In addition, we present methods of constructing an interpolation *xyB*-spline curve and an *xyB*-spline curve which is tangent to the given control polygon. The generation of tensor product surfaces by these new spline curves is straightforward. Many properties of the curves can be easily extended to the surfaces. The new surfaces can exactly represent the rotation surfaces as well as the surfaces with elliptical or circular sections.

1. Introduction

As a unified mathematic model with many desirable properties, *B*-splines are popularly applied in modeling free-form curves and surfaces. However, there still exist several limitations of the *B*-spline model, which limit its applications. First, when the knot sequences are specified, the position of *B*-spline curves and surfaces is fixed relative to their control points. Second, they cannot exactly represent conics (except parabolas) and some transcendental curves such as the cycloid and the helix, which are often used in engineering. Finally, when using *B*-splines to construct interpolation curves, we must solve a system of equations to get the control points. Although NURBS can overcome the first two shortcomings of *B*-splines to a certain extent, it also fails to represent some remarkable transcendental curves, and it also exist the third limitation as the *B*-spline model. Furthermore, the NURBS model suffers from several new drawbacks due to the relative complexity of rational basis functions. For example, the rational form may be unstable, and derivatives and integrals are hard to compute. In order to avoid the inconveniences of the *B*-spline model and the NURBS model, finding new spline models seems to be the only way.

Aiming at the drawbacks of *B*-splines, a number of new modeling methods have been presented in recent years. In order to enhance the flexibility of *B*-spline models, scholars put forward many curves and surfaces with shape parameters through incorporating parameters into the classical basis functions, where the parameters can adjust the shape of the curves and surfaces without changing the control points (cf. [1–3]). In order to extend the shape representation range of the *B*-spline model, scholars put forth several new models in non-polynomial spaces. For instance, Zhang (cf. [4–6]) constructed *C*-Bezier and *CB*-spline curves and surfaces in the space $\{1, t, \cos t, \sin t\}$. Mainar and Pena (cf. [7]), Chen and Wang (cf. [8]) defined *C*-Bezier curves of high order in the space $\{1, t, \ldots, t^{k-3}, \cos t, \sin t\}$. In the same space, nonuniform algebraic–trigonometric *B*-splines (NUTA splines) were constructed by Wang and Chen (cf. [9]). The *C* style curves using sin and cos can represent the ellipse, the helix, and the cycloid exactly. Lü et al. (cf. [10]) proposed uniform hyperbolic polynomial *B*-

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splines curves in the space $\{1, t, \dots, t^{k-3}, \cos ht, \sin ht\}$. Li and Wang (cf. [11]) extended these hyperbolic splines to the case of nonuniform knot vector. The *H* style curves using $\sin h$ and $\cos h$ admit exact representations for the hyperbola and the catenary. Zhang and Krause (cf. [12]) unified *CB*-splines and *HB*-splines into *FB*-splines (Functional *B*-splines) by a unified basis. Again, Zhang et al. (cf. [13]) unified *C*-curves and *H*-curves into *F*-curves by extending the calculation to complex numbers. Later, Wang and Fang (cf. [14]) unified and extended polynomial, trigonometric and hyperbolic splines by a new kind of spline (*UE*-spline for short) over the space $\{\cos \omega t, \sin \omega t, 1, t, \dots, t^l, \dots\}$. In order to avoid complex computation when using *B*-splines to construct interpolation curves, many scholars presented new interpolation schemes. For example, Tai and Loe (cf. [15]), Pan and Wang (cf. [16]), Tai and Wang (cf. [17]) presented interpolation methods without solving a global system of equations using singular blending. The basic idea of these papers is to blend NURBS curves or *B*-spline curves with a singularly parameterized sequence of connected line segments.

Many new models which improved *B*-splines from different aspects have been proposed. Each model has its own merits. But there is no model which can overcome all the shortcomings mentioned above at present. This paper tries to overcome these shortcomings. We construct a new kind of spline curve with the same structure as the usual cubic *B*-spline curves. The new curves are shape adjustable, and they can express some conics as well as some transcendental curves precisely. Furthermore, they not only can approach given points, but can also interpolate given points automatically. In addition, in modeling curves and surfaces, we often encounter problems such as: the tangent lines of the outline are known, how to find a better curve to approximate them. That is, how to construct curves which are tangential to the given polygons (cf. [18–20]). As an application, this paper also discusses how to construct *xyB* curves which are tangential to the given polygons. According to the method given in this paper, all control points of the *xyB* curves can be calculated simply by the vertices of the given polygon.

The rest of this paper is organized as follows. Section 2 gives the definition and properties of *xyB* functions. In Section 3, we define *xyB* curves and list some properties of them. We give the representations of some known curves in Section 4. In Section 5, we discuss how to construct *xyB* curves with given tangent polygon. We describe the method for automatically constructing interpolatory spline curves with *xyB* functions in Section 6. In Section 7, we discuss tensor product *xyB* surfaces. A short conclusion is given in Section 8.

2. The *xyB* functions

2.1. The definition of xyB functions

Definition 2.1. We define the algebraic–trigonometric blending functions with two parameters *x* and *y* (*xyB* functions for short) as follows:

$$\begin{cases} B_0(t) = \dfrac{1}{2} - \dfrac{4x+y}{12} + \dfrac{x-1}{\pi}(t+C) + \dfrac{y-8x}{24}S + \dfrac{x+y}{6}S^2 - \dfrac{y}{8}S^3 \\ B_1(t) = \dfrac{4x+y}{12} + \dfrac{1-x}{\pi}(t-S) + \left[\dfrac{2(1-x)}{\pi} + \dfrac{8x-y}{24}\right]C - \dfrac{x+y}{6}S^2 + \dfrac{y}{8}C^3 \\ B_2(t) = \dfrac{1}{2} - \dfrac{4x+y}{12} + \dfrac{x-1}{\pi}(t+C) + \left[\dfrac{2(1-x)}{\pi} + \dfrac{8x-y}{24}\right]S + \dfrac{x+y}{6}S^2 + \dfrac{y}{8}S^3 \\ B_3(t) = \dfrac{4x+y}{12} + \dfrac{1-x}{\pi}(t-S) + \dfrac{y-8x}{24}C - \dfrac{x+y}{6}S^2 - \dfrac{y}{8}C^3 \end{cases}, \quad t \in \left[0, \frac{\pi}{2}\right] \tag{1}$$

where $x, y \in R$, $S := \sin(t)$, $C := \cos(t)$.

Remark 2.1. To get these functions, we constructed two groups of bases before. The first group is defined over the space $\{1, t, \cos t, \sin t\}$. The curves defined based on it can exactly represent the ellipse, the circle, the sine curve, the cosine curve, the cycloid, and the helix. However, the shape of these curves is fixed relative to their control points, and they are only C^2 continuous. The second group is defined over the space $\{1, \sin t, \cos t, \sin^2 t, \sin^3 t, \cos^3 t\}$. The curves defined based on it are shape adjustable, and they have better continuity. In general, they are C^3 continuous, and they are C^5 continuous with proper parameters. However, these curves cannot express conics and transcendental curves except for the ellipse. In order to get the curves with better properties, we made a linear combination of the two groups of bases, and made a substitution to simplify the expression of the functions. Thus, we get *xyB* functions.

Definition 2.2. When the parameters of *xyB* functions take values in the ranges $x \in [0, 1]$, $y \in [0, \frac{4}{5}]$, or $x \in [-3, 0]$, $y \in [0, \frac{3}{2}]$, we call $B_i(t)$ $(i = 0, 1, 2, 3)$ *xyB* bases.

2.2. Properties of xyB functions

(1) *Monotonicity*: For fixed $t \in [0, \frac{\pi}{2}]$, $B_0(t)$ and $B_3(t)$ are monotonically decreasing about *x* and *y*.