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Asymptotic behaviour of Laguerre–Sobolev-type orthogonal polynomials. A nondiagonal case

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To Professor Adhemar Bultheel on the occasion of his 60th birthday

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ABSTRACT

In this paper we study the asymptotic behaviour of polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \quad \alpha > -1,$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}, \qquad \mathbb{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}, \qquad \mathbb{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

and A is a positive semidefinite matrix.

We will focus our attention on their outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler–Heine formula for the rescaled polynomials.

1. Introduction

Orthogonal polynomials with respect to a Sobolev-type inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x) + \mathbb{P}(c)^t A\mathbb{Q}(c), \tag{1}$$

where $d\mu$ is a nontrivial probability measure supported on the real line, $A \in \mathbb{R}^{(k,k)}$ is a positive semidefinite matrix, p, q are polynomials with real coefficients, and $\mathbb{Q}(c) = \left(q(c), q'(c), \dots, q^{(k-1)}(c)\right)^t$ have been introduced in [1].

When $A = \operatorname{diag}(M_0, M_1, \dots, M_{k-1})$, the so-called diagonal Sobolev-type case, many researchers were interested in the analytic properties of the polynomials orthogonal with respect to (1). In particular, Koekoek [2] studied the second order linear differential equation satisfied by such orthogonal polynomials when $d\mu = x^\alpha e^{-x}dx$, $\alpha \geq 1$, and $c = 0$. They also satisfy a higher order recurrence relation as well as they can be represented as hypergeometric functions.

Later on, when $k = 2$ and $M_0, M_1 > 0$, in [3] the authors focus the attention in the location of the zeros of such orthogonal polynomials that are called Laguerre–Sobolev-type orthogonal polynomials. Finally, the analysis of their asymptotic properties was done in [4] as well in [5].

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On the other hand, when $k \geq 2$ if $d\mu = x^{\alpha}e^{-x}dx$, $c = 0$, and $M_0 = M_1 = \cdots = M_{k-2} = 0$, $M_{k-1} > 0$ then the same analog problems were studied in [6] in the framework of the zero distribution. From an algebraic point of view and for more general measures, in [7] the authors deal with representations of Sobolev-type orthogonal polynomials in terms of the polynomials orthogonal with respect to the measure μ assuming the same constraints for the inner product (1) as above.

The first situation of a nondiagonal Sobolev-type inner product like (1) was considered in [8]. Here the authors deal with the measure $d\mu = e^{-x^2}dx$ supported on $\mathbb{R}$, $c = 0$, and $k = 2$. In particular, they analyze scaled asymptotics for the corresponding orthogonal polynomials (Mehler–Heine formulas) and, as a consequence, the asymptotic behaviour of their zeros follows.

Taking into account that generalized Hermite polynomials appear as a consequence of the symmetrization process for Laguerre orthogonal polynomials [9–11] it seems to be very natural to analyze polynomial sequences orthogonal with respect to the inner product (1) when $d\mu = x^{\alpha}e^{-x}dx$, $A \in \mathbb{R}^{(k,k)}$ is a nondiagonal positive semidefinite matrix with $k \geq 2$, and $c = 0$.

In this contribution we focus our attention in the case $k = 2$. Thus we generalize some previous results from the diagonal case (see [4,12,3]). The structure of the manuscript is the following. In Section 2 we present the basic background about the properties of classical Laguerre polynomials which will be needed along the paper. Section 3 deals with the asymptotic properties of the Laguerre–Sobolev-type polynomials, orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \quad \alpha > -1,$$

where $A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$ is a positive semidefinite matrix and we denote $\mathbb{Q}(0) = \left(q(0), q'(0)\right)^{t}$. We obtain the outer relative asymptotics of these polynomials in terms of Laguerre polynomials and a Mehler–Heine-type formula as well as the behaviour of the Sobolev norm of the monic Laguerre–Sobolev-type orthogonal polynomials.

2. Preliminaries

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be the linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\left\langle \mu, x^n\right\rangle = \mu_n, \quad n = 0, 1, 2, \ldots$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Furthermore μ_n is the *n-th moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.
(ii) $\langle \mu, P_n(x)P_m(x)\rangle = 0$, $m \neq n$.
(iii) $\left\langle \mu, P_n^2(x)\right\rangle \neq 0$, $n = 0, 1, 2, \ldots$.

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.*

The next theorem, whose proof appears in [10], gives a necessary and sufficient condition for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 ([10]). *Let μ be the moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are nonsingular.*

A moment functional such that there exists the corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* [10].

The proof of the next proposition can be founded in [9,10,13,14,11].

Proposition 1 (*The Christoffel–Darboux Formula*). *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials. If we denote the nth kernel polynomial by*

$$K_n(x, y) = \sum_{j=0}^{n} \frac{P_j(y)P_j(x)}{\left\langle \mu, P_j^2\right\rangle},$$

then, for every $n \in \mathbb{N}$,

$$K_n(x, y) = \frac{1}{\left\langle \mu, P_n^2\right\rangle} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}. \tag{2}$$

Using the following notation for the partial derivatives of the kernel $K_n(x, y)$

$$\frac{\partial^{j+k}\left(K_n(x, y)\right)}{\partial^j x\partial^k y} = K_n^{(j,k)}(x, y),$$