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The role of coefficients of a general SPDE on the stability and convergence of a finite difference method

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ABSTRACT

In this paper for the approximate solution of stochastic partial differential equations (SPDEs) of Itô-type, the stability and application of a class of finite difference method with regard to the coefficients in the equations is analyzed. The finite difference methods discussed here will be either explicit or implicit and a comparison between them will be reported. We prove the consistency and stability of these methods and investigate the influence of the multiplier (particularly multiplier of the random noise) in mean square stability. From stochastic version of Lax–Richtmyer the convergence of these methods under some conditions are established. Numerical experiments are included to show the efficiency of the methods.

1. Introduction

In recent years, a great deal of concern has been raised regarding the study of stochastic partial differential equations as an important area of research. Many phenomena in science and engineering that may have been modeled by deterministic partial differential equations, have some uncertainty, due to the existence of different stochastic perturbations. Therefore to represent a more accurate detail of behavior of such phenomena they usually should be modeled by SPDEs. SPDEs have many applications in continuum physics [1,2], finance, for example for contingent claim, bond pricing problem, interest rate of option and forward caps [3–6]. SPDEs have been studied theoretically in [7–10]. Several authors investigated numerical solutions of stochastic PDEs. Some authors used finite element approximation, e.g, [11,12]. Some others like Gyöngy, Gaines, Davie [13–15], used finite difference methods. They approximate quasi-linear parabolic SPDEs by substituting the space variable derivative, and obtain some results on the convergence of the resulting difference equations. Yoo presents the semi-discretization of SPDEs by finite difference methods, and analyzes the sup-norm error and the rate of convergence of the approximation method [16]. Roth used an explicit finite difference method to approximate the solutions of some stochastic hyperbolic equations [17,18]. In this paper we use an explicit and also an implicit finite difference method to fully discretize the following SPDE to investigate the effect of coefficients in the convergence and stability of the approximate solution

$$u_t(x,t) + au_{xx}(x,t) + bu_x(x,t) + cu(x,t) + (du_x(x,t) + \gamma u(x,t))\dot{W}(t) = 0 \tag{1.1}$$

where $\dot{W}(t)$ is a random process which is related to the Brownian motion $W(t)$ where $\dot{W}(t) = \frac{\partial W}{\partial t}(t)$.

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We know that this is not an ordinary derivative. This is a distributional derivative because Brownian motion is nowhere differentiable. Random process $W(t)$ is Gaussian with zero mean. This equation can be read as

$$u(x,t)-u(x,0)+\int_0^t (au_{xx}(x,s)+bu_x(x,s)+cu(x,s))\mathrm{d}s+\int_0^t (du_x(x,s)+\gamma u(x,s))\mathrm{d}W(s)=0,$$

where a, b, c, d, γ are constants. The stochastic integral is the Itô-Integral with respect to $\mathbb{R}^1$-valued Wiener process $(W(t), F_t)_{t\in[0,T]}$ defined on a complete probability space (Ω, F, P), adapted to the standard filtration $(F_t)_{t\in[0,T]}$. Different values of the coefficients of the above equation correspond to different applications. For example, the case $a=1$, $b=c=0$, represents a parabolic stochastic heat equation with many applications in engineering, $a=0$ corresponds to a hyperbolic stochastic equation with applications in finance mathematics, say, interest rate of options [17]. We investigate consistency and stability of two proposed finite difference methods for SPDE (1.1) in a more general sense. The paper is organized as follows. In Section 2 we use an explicit finite difference method for Eq. (1.1), and investigate consistency and stability of the resulting stochastic difference scheme (SDS). In Section 3 we use the Crank–Nicolson method and investigate the stability of this method. This analysis clearly illustrates different levels of the influence of the coefficients in the equation. Finally in the last section numerical examples are presented and compared with some analytical solutions.

2. An explicit finite difference method

Consider the following SPDE

$$u_t(x,t)+au_{xx}(x,t)+bu_x(x,t)+cu(x,t)+(du_x(x,t)+\gamma u(x,t))\dot{W}(t)=0 \tag{2.1}$$

with initial condition $u(x,0)=f(x),\ 0\le x\le 1$ and boundary conditions

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T.$$

Random variable u_i^j will be used to denote the approximation solution of (2.1) at the point $(i\Delta x, j\Delta t)$. We approximate $u_t(x,t)$ with

$$u_t(x,t)\approx\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} \tag{2.2}$$

and $u_x(i\Delta x, j\Delta t)$, $u_{xx}(i\Delta x, j\Delta t)$ with

$$u_x(i\Delta x,j\Delta t)\approx\frac{u_{i+1}^j-u_i^j}{\Delta x},\qquad u_{xx}(i\Delta x,j\Delta t)\approx\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}. \tag{2.3}$$

Substituting these partial derivatives in Eq. (1.1) the following explicit finite difference approximation is obtained

$$\frac{u_i^{j+1}-u_i^j}{\Delta t}+a\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}+b\frac{u_{i+1}^j-u_i^j}{\Delta x}+cu_i^j+\left(d\frac{u_{i+1}^j-u_i^j}{\Delta x}+\gamma u_i^j\right)\frac{W((j+1)\Delta t)-W(j\Delta t)}{\Delta t}=0. \tag{2.4}$$

Eq. (2.4) can be written as

$$\begin{aligned}u_i^{j+1} &= \left(-a\frac{\Delta t}{(\Delta x)^2}\right)u_{i-1}^j+\left(1+2a\frac{k}{(\Delta x)^2}+b\frac{\Delta t}{\Delta x}-c\Delta t\right)u_i^j+\left(-a\frac{\Delta t}{(\Delta x)^2}-b\frac{\Delta t}{\Delta x}\right)u_{i+1}^j\\ &\quad-\left(\frac{d}{\Delta x}(u_{i+1}^j-u_i^j)+\gamma u_i^j\right)(W((j+1)\Delta t)-W(j\Delta t)),\end{aligned} \tag{2.5}$$

where by substituting $R_1=a\frac{\Delta t}{(\Delta x)^2}$, $R_2=b\frac{\Delta t}{\Delta x}$ one obtains

$$\begin{aligned}u_i^{j+1} &= (-R_1)u_{i-1}^j+(1+2R_1+R_2-c\Delta t)u_i^j+(-R_1-R_2)u_{i+1}^j\\ &\quad-\left(\frac{d}{\Delta x}(u_{i+1}^j-u_i^j)+\gamma u_i^j\right)(W((j+1)\Delta t)-W(j\Delta t)),\end{aligned} \tag{2.6}$$

which is known as the corresponding stochastic finite difference scheme.