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On Taylor-series expansion methods for the second kind integral equations

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ABSTRACT

In this paper, we comment on the recent papers by Yuhe Ren et al. (1999) [1] and Maleknejad et al. (2006) [7] concerning the use of the Taylor series to approximate a solution of the Fredholm integral equation of the second kind as well as a solution of a system of Fredholm equations. The technique presented in Yuhe Ren et al. (1999) [1] takes advantage of a rapidly decaying convolution kernel $k(|s-t|)$ as $|s-t|$ increases. However, it does not apply to equations having other types of kernels. We present in this paper a more general Taylor expansion method which can be applied to approximate a solution of the Fredholm equation having a smooth kernel. Also, it is shown that when the new method is applied to the Fredholm equation with a rapidly decaying kernel, it provides more accurate results than the method in Yuhe Ren et al. (1999) [1]. We also discuss an application of the new Taylor-series method to a system of Fredholm integral equations of the second kind.

1. Introduction

In paper [1], a Taylor-series expansion method to approximate a solution of a class of Fredholm integral equations of the second kind was considered. The Fredholm equation of the second kind takes the following form:

$$x(s) - \int_0^1 k(s,t)x(t)\mathrm{d}t = y(s), \quad 0 \le s \le 1, \tag{1.1}$$

where it is assumed that 1 is not the eigenvalue of the operator

$$Tx(s) \equiv \int_0^1 k(s,t)x(t)\mathrm{d}t.$$

The kernel $k(s,t) = k(|s-t|)$ is assumed to be continuous in $I \equiv [0,1]$ and decreases as $|s-t|$ increases from zero or $k(s,t) = a(s,t)\kappa(s-t)$ with a is continuous for $s,t \in I$ and $\kappa(s-t) = O(|s-t|^{-\alpha})$, $0 < \alpha < 1$. Our numerical experiments indicate that the Taylor method introduced in [1] is effective under the first assumption, particularly, in the case that the rate of decay to 0 of $k(|s-t|)$ is sufficiently large, but as was reported in [1], the technique does not perform well under the second assumption of weakly singular kernel. For numerical solutions of weakly singular Fredholm equations, the present authors suggest that it is better to use the standard Galerkin method or the collocation method to obtain numerical solutions which exhibit optimal order of convergence; see, e.g., [2–5] and the references cited within. As described in [1], the Fredholm equations of the second kind play an important role in many physical applications which include potential theory and Dirichlet problems, particle transport problems of astrophysics and radiative heat transfer problems. A reader may consult the references provided in [1] for these applied problems.

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The first step in the Taylor expansion method presented in [1] is to write

$$x(t) \approx x(s) + x'(s)(t-s) + \cdots + \frac{1}{n!}x^{(n)}(s)(t-s)^n. \tag{1.2}$$

Substituting (1.2) for $x(t)$ in the integral in (1.1), we obtain

$$\left[1 - \int_0^1 k(s,t)\mathrm{d}t\right]x(s) - \left[\int_0^1 k(s,t)(t-s)\mathrm{d}t\right]x'(s) - \cdots - \left[\frac{1}{n!}\int_0^1 k(s,t)(t-s)^n\mathrm{d}t\right]x^{(n)}(s) \approx y(s), \quad 0 < s < 1. \tag{1.3}$$

Eq. (1.3) represents an nth order linear ordinary differential equation with variable coefficients. However, in order to carry out the solution process, it is necessary that an appropriate number of boundary conditions be introduced. These boundary conditions may be found from an actual experiment, but this is not always possible. To circumvent this problem, paper [1] proceeds as follows. First, differentiating (1.1) n times, one obtains

$$\begin{aligned} x'(s) - \int_0^1 k'_s(s,t)x(t)\mathrm{d}t &= y'(s) \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s,t)x(t)\mathrm{d}t &= y^{(n)}(s), \end{aligned} \tag{1.4}$$

where $k_s^{(i)}(s,t) = \partial^{(i)}k(s,t)/\partial s^i$, $i = 1, \ldots, n$. Next, $x(t)$ is replaced by $x(s)$ to obtain, for $0 < s < 1$,

$$\begin{aligned} x'(s) - \int_0^1 k'_s(s,t)\mathrm{d}t\, x(s) &= y'(s) \\ &\vdots \\ x^{(n)}(s) - \int_0^1 k_s^{(n)}(s,t)\mathrm{d}t\, x(s) &= y^{(n)}(s). \end{aligned} \tag{1.5}$$

The step taken in (1.5) characterizes the Taylor technique of [1] and it is justified by the first assumption above that $k(|s-t|)$ decays rapidly as $|s-t|$ increases. Eq. (1.3) together with equations in (1.5) can be used to solve for $x, x', \ldots, x^{(n)}$. More specifically, we solve the following system of linear equations for $x(s), x'(s), \ldots, x^{(n)}(s)$.

$$\begin{bmatrix} 1 - \int_0^1 k(s,t)\mathrm{d}t & -\int_0^1 k(s,t)(t-s)\mathrm{d}t & \cdots & -\frac{1}{n!}\int_0^1 k(s,t)(t-s)^n\mathrm{d}t \\ -\int_0^1 k'_s(s,t)\mathrm{d}t & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\int_0^1 k_s^{(n)}(s,t)\mathrm{d}t & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x(s) \\ x'(s) \\ \vdots \\ x^{(n)}(s) \end{bmatrix} = \begin{bmatrix} y(s) \\ y'(s) \\ \vdots \\ y^{(n)}(s) \end{bmatrix}. \tag{1.6}$$

A step taken in (1.5) needs some investigation. In (1.1), solution x is expanded by the Taylor series to its nth degree and yet in (1.5) $x(t)$ is replaced by its constant approximation $x(s)$. Despite this, under a fast decaying kernel, solution of (1.6) produces reasonably accurate approximation as reported in the paper [1]. In the next section, we propose to retain the nth degree expansion of Taylor series for $x(t)$ in (1.5). Of course, this requires additional computations, but the additional cost is justified by the fact that the new method applies not only to a much wider class of Fredholm equations but also produces more accurate approximations. Also, the present method computes $x(s), x'(s), \ldots, x^{(n)}(s)$ all within the same accuracy in the same solution process.

2. Modified Taylor-series method

Here, we begin by replacing each $x(t)$ in (1.5) by the right side of (1.2) to obtain

$$\begin{aligned} -\int_0^1 k'_s(s,t)\mathrm{d}t\, x(s) - \left[\int_0^1 k'_s(s,t)(t-s)\mathrm{d}t - 1\right]x'(s) - \cdots - \frac{1}{n!}\int_0^1 k'_s(s,t)(t-s)^n\mathrm{d}t\, x^{(n)}(s) &= y'(s) \\ &\vdots \\ -\int_0^1 k_s^{(n)}(s,t)\mathrm{d}t\, x(s) - \int_0^1 k_s^{(n)}(s,t)(t-s)\mathrm{d}t\, x'(s) - \cdots - \left[\frac{1}{n!}\int_0^1 k_s^{(n)}(s,t)(t-s)^n\mathrm{d}t - 1\right]x^{(n)}(s) &= y^{(n)}(s). \end{aligned} \tag{2.1}$$