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The polynomial Fourier transform with minimum mean square error for noisy data

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ABSTRACT

In 2006, Naoki Saito proposed a Polyharmonic Local Fourier Transform (PHLFT) to decompose a signal $f \in L^2(\Omega)$ into the sum of a *polyharmonic component* u and a *residual* v, where Ω is a bounded and open domain in $\mathbb{R}^d$. The solution presented in PHLFT in general does not have an error with minimal energy. In resolving this issue, we propose the least squares approximant to a given signal in $L^2([-1, 1])$ using the combination of a set of algebraic polynomials and a set of trigonometric polynomials. The maximum degree of the algebraic polynomials is chosen to be small and fixed. We show in this paper that the least squares approximant converges uniformly for a Hölder continuous function. Therefore Gibbs phenomenon will not occur around the boundary for such a function. We also show that the PHLFT converges uniformly and is a near least squares approximation in the sense that it is arbitrarily close to the least squares approximant in L^2 norm as the dimension of the approximation space increases. Our experiments show that the proposed method is robust in approximating a highly oscillating signal. Even when the signal is corrupted by noise, the method is still robust. The experiments also reveal that an optimum degree of the trigonometric polynomial is needed in order to attain the minimal l^2 error of the approximation when there is noise present in the data set. This optimum degree is shown to be determined by the intrinsic frequency of the signal. We also discuss the energy compaction of the solution vector and give an explanation to it.

1. Introduction

In 2006, Naoki Saito proposed a Polyharmonic Local Sine Transform (PHLST) [1] in an attempt to develop a local Fourier analysis and synthesis method without encountering the infamous Gibbs phenomenon. PHLST is also used to resolve several problems occurring in the Local Trigonometric Transforms (LTTs) of Coifman and Meyer [2] and Malvar [3,4], such as the overlapping windows and the slopes of the bell functions (see [1] for the details on how PHLST resolves these problems).

Let us pause for a moment to define some notations. To index infinitely countable sets, we adopt the following standard conventions: let $\mathbb{N}$ be the set of natural numbers, and the set $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. To enumerate finite sets, we define $\mathbb{Z}_k := \{0, 1, \ldots, k-1\}$, and $\mathbb{N}_k := \{1, 2, \ldots, k\}$.

PHLST first segments a given function (or input data) $f(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega \subset \mathbb{R}^d$ supported on an open and bounded domain Ω into a set of disjoint blocks $\{\Omega_j : j \in \mathbb{Z}_M\}$ for a positive integer M such that $\overline{\Omega} = \bigcup_{j\in\mathbb{Z}_M} \overline{\Omega}_j$. Denote by f_j the restriction of the function f to $\overline{\Omega}_j$, i.e., $f_j = \chi_{\overline{\Omega}_j} f$, where $\chi_{\overline{\Omega}_j}$ is the characteristic function on the set $\overline{\Omega}_j$, $j \in \mathbb{Z}_M$. Then PHLST decomposes

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each f_j into two components as $f_j = u_j + v_j$. The components u_j and v_j are referred to as the *polyharmonic component* and the *residual*, respectively. The polyharmonic component is obtained by solving the following *polyharmonic equation*:

$$\Delta^m u_j = 0 \quad \text{in } \Omega_j,\ m \in \mathbb{N} \tag{1.1}$$

with a set of given boundary values and normal derivatives

$$\frac{\partial^{q_\ell} u_j}{\partial \nu^{q_\ell}} = \frac{\partial^{q_\ell} f}{\partial \nu^{q_\ell}} \quad \text{on } \partial\Omega_j,\ \ell \in \mathbb{Z}_m, \tag{1.2}$$

where $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$ is the Laplace operator in $\mathbb{R}^d$. The natural number m is called the degree of polyharmonicity, and q_ℓ is the order of the normal derivative. These boundary conditions (1.2) enforce that the solution u_j interpolates the function values and the normal derivatives of orders $q_1, \ldots, q_{m-1}$ of the original signal f along the boundary $\partial\Omega_j$. The parameter q_0 is normally set to 0, which means that $u_j = f$ on the boundary $\partial\Omega_j$, i.e., the Dirichlet boundary condition. If the blocks $\Omega_j, j \in \mathbb{Z}_M$, are all rectangles (of possibly different sizes), PHLST sets $q_\ell = 2\ell$, i.e., only normal derivatives of *even orders* are interpolated. It is not necessary to match normal derivatives of odd orders when the blocks Ω_j's are rectangular domains. This is because the Fourier sine series of the residual v_j is equivalent to the complex Fourier series of the periodized v_j after odd reflection with respect to the boundary $\partial\Omega_j$, hence the continuity of the normal derivatives of odd orders (up to order $2m-1$) is automatically guaranteed. Thanks to these boundary conditions, the residual component can be expanded into a Fourier sine series without facing the Gibbs phenomenon, and the Fourier sine expansion coefficients of the residual v_j decay rapidly, i.e., in the order $O(\|\boldsymbol{k}\|^{-2m-1})$, provided that there is no other intrinsic singularity in the domain Ω_j, where $\boldsymbol{k}$ is the frequency index vector.

In our joint work [5], we implemented PHLST up to polyharmonicity of degree 5. The corresponding algorithm is called PHLST5. In that work, we derived a fast algorithm to compute a 5th degree polyharmonic function that satisfies certain boundary conditions. Although the Fourier sine coefficients of the residual of PHLST5 possess the same decaying rate as in LLST (Laplace Local Sine Transform, the simplest version of PHLST with polyharmonicity of degree 1), by using additional information of first order normal derivative from the boundary, the blocking artifacts are largely suppressed in PHLST5 and the residual component becomes much smaller than that of LLST. Therefore PHLST5 provides a better approximation result. Due to the difficulty of estimating higher order derivatives, we consider PHLST5 as the practical limitation of implementing PHLST with higher degree polyharmonicity.

Soon after developing PHLST, N. Saito and K. Yamatani extended it to the *Polyharmonic Local Cosine Transform* (PHLCT) [6]. The PHLCT allows the Fourier cosine coefficients of the residual decay in the order $O\left(\|\boldsymbol{k}\|^{-2m-2}\right)$ by setting $q_\ell = 2\ell + 1,\ \ell \in \mathbb{Z}_m$ in the boundary conditions (1.2) and by introducing an appropriate source term on the right-handed side of the polyharmonic equation (1.1). In that work, an efficient algorithm was developed to improve the quality of images already severely compressed by the popular JPEG standard, which is based on Discrete Cosine Transform (DCT).

Finally, N. Saito introduced the *Polyharmonic Local Fourier Transform* (PHLFT) [1] by setting $q_\ell = \ell, \ell \in \mathbb{Z}_m$ in Eq. (1.2) and by replacing the Fourier sine series with the complex Fourier series in expanding the v_j components. With some sacrifice of the decay rate of the expansion coefficients, i.e., of order $O\left(\|\boldsymbol{k}\|^{-m-1}\right)$ instead of order $O\left(\|\boldsymbol{k}\|^{-2m-1}\right)$ or of order $O\left(\|\boldsymbol{k}\|^{-2m-2}\right)$, PHLFT allows one to compute local Fourier magnitudes and phases without facing the Gibbs phenomenon. PHLFT also can capture the important information of orientation much better than PHLST and PHLCT. Moreover, it is fully invertible and should be useful for various filtering, analysis, and approximation purposes.

Although the Fourier coefficients of the residual v decay rapidly, it is virtually useless for the purpose of approximation. *Therefore, in practice we shall not only seek fast decaying rate of the Fourier coefficients of the residual v, but also a residual v of a small energy*. However, the residual v in PHLST (or in PHLCT or PHLFT) in general does not necessarily have minimal energy. In resolving this issue, we propose the least squares approximant to a given signal using the combination of a set of algebraic polynomials and a set of trigonometric polynomials. The maximum degree of the algebraic polynomials is chosen to be small and fixed. We show in this paper that the least squares approximant converges uniformly for a Hölder continuous function. Therefore the Gibbs phenomenon will not occur on the boundary for such functions. We also show that the PHLST (or PHLCT, PHLFT) converges uniformly and is a near least squares approximation in the sense that it is arbitrarily close to the least squares approximant in the L^2 norm as the dimension of the approximation space increases. Our experiments show the proposed method is robust in approximating a highly oscillating signal. Even when the signal is corrupted by noise, the method is still robust. The experiments also reveal that an optimum degree of trigonometric polynomial is needed in order to attain minimal l^2 error of the approximation when there is noise present in the data set. This optimum degree is shown to be determined by the intrinsic frequency of the signal. We also discuss the energy compaction of the solution vector and give an explanation to it.

2. Problem formulation and characterization of the solution

Let f be a noise corrupted and finite-energy signal on the interval $J := [-1, 1]$, that is $f \in L^2(J)$. The L^2 norm of a function $f \in L^2(J)$ is denoted by $\|f\|$, that is $\|f\|^2 := \int_J |f(x)|^2\, dx$. Other norms used in this paper will be indicated by the appropriate subscripts.