Journal of Computational and Applied Mathematics 234 (2010) 365–374

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On constructing new expansions of functions using linear operators

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ARTICLE INFO

Article history:
Received 24 August 2008

MSC:
42C15
42C20
42B05
42A16
41A58

Keywords:
Mixed Taylor–Fourier and mixed orthogonal polynomial expansions
Mixed Taylor expansions
Inverse Laplace transforms
Linear operators
Functional equations
Finite approximations

ABSTRACT

Let T, U be two linear operators mapped onto the function f such that $U(T(f)) = f$, but $T(U(f)) \neq f$. In this paper, we first obtain the expansion of functions $T(U(f))$ and $U(T(f))$ in a general case. Then, we introduce four special examples of the derived expansions. First example is a combination of the Fourier trigonometric expansion with the Taylor expansion and the second example is a mixed combination of orthogonal polynomial expansions with respect to the defined linear operators T and U. In the third example, we apply the basic expansion $U(T(f)) = f(x)$ to explicitly compute some inverse integral transforms, particularly the inverse Laplace transform. And in the last example, a mixed combination of Taylor expansions is presented. A separate section is also allocated to discuss the convergence of the basic expansions $T(U(f))$ and $U(T(f))$.

1. Introduction

Let $\mathbf{S} = \{S_i(.)\}_{i=0}^{\infty}$ be an infinite set of all *linear* arbitrary functionals [1,2], which are defined on a linear vector space and $\boldsymbol{\Phi} = \{\Phi_i(x)\}_{i=0}^{\infty}$ be a certain set of basis functions. In this case, it is clear that we have

$$S_j\left(\sum_{i=0}^{m} a_i\Phi_i(x)\right) = \sum_{i=0}^{m} a_i S_j\left(\Phi_i(x)\right) \quad \text{for any } j = 0, 1, \ldots, \tag{1}$$

where $\{a_i\}_{i=0}^{m}$ are arbitrary constants.

We start our discussion with a main problem: Suppose that the following equality is given

$$f(x) = \sum_{i=0}^{n} S_i(f)\Phi_i(x), \tag{2}$$

where $\{S_i(f)\}_{i=0}^{n} \subset \mathbf{S}$ and $\{\Phi_i(x)\}_{i=0}^{n} \subset \boldsymbol{\Phi}$.

In general, two different viewpoints can be considered for equality (2). Either it is a *functional equation* to be solved, e.g. the following equation

$$f(x) = f(0) + f'(2)x^2 + \left(\int_0^1 f(x)\mathrm{d}x\right)x^3, \tag{2.1}$$

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doi:10.1016/j.cam.2009.12.030

or it is a *finite approximation* for $f(x)$ like:

$$f(x)=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}, \tag{2.2}$$

which is in fact a Taylor expansion of order 3 at $x=0$.

Now, the main problem is how to recognize that (2) is a functional equation or is a finite approximation for $f(x)$. For instance, compare two given equalities in (2.1) and (2.2). Although both of them are just special cases of equality (2), how can one find out that (2.1) is a functional equation while (2.2) is a finite approximation for $f(x)$? This is an important question that should be answered. Before responding to this question, we should note that both above-mentioned cases lead to only one result, i.e. determining the unknown functionals $\{S_i(f)\}_{i=0}^n$ in (2) appropriately.

To solve the problem, first note that since $\{S_i(f)\}_{i=0}^n$ are all *linear* functionals, taking $S_0, S_1, \ldots$ and S_n from both sides of (2) respectively yields

$$\begin{cases} S_0(f)=S_0\left(\sum_{i=0}^n S_i(f)\Phi_i(x)\right)=\sum_{i=0}^n S_i(f)S_0(\Phi_i(x)),\\ S_1(f)=S_1\left(\sum_{i=0}^n S_i(f)\Phi_i(x)\right)=\sum_{i=0}^n S_i(f)S_1(\Phi_i(x)),\\ \vdots\\ S_n(f)=S_n\left(\sum_{i=0}^n S_i(f)\Phi_i(x)\right)=\sum_{i=0}^n S_i(f)S_n(\Phi_i(x)). \end{cases} \tag{3}$$

If the above expressions constitute a linear homogeneous system, then (2) can be interpreted as a functional equation. In this case, (3) takes the matrix form

$$MX=\begin{bmatrix} S_0(\Phi_0)-1 & S_0(\Phi_1) & \cdots & S_0(\Phi_n)\\ S_1(\Phi_0) & S_1(\Phi_1)-1 & \cdots & S_1(\Phi_n)\\ \vdots & \vdots & \vdots & \vdots\\ S_n(\Phi_0) & S_n(\Phi_1) & \cdots & S_n(\Phi_n)-1 \end{bmatrix}\begin{bmatrix} S_0(f)\\ S_1(f)\\ \vdots\\ S_n(f)\end{bmatrix}=0, \tag{3.1}$$

where M denotes the coefficients matrix and X is the unknown functionals vector.

After solving the above linear system with respect to one of the pre-assigned functionals (provided that $\det M=0$), the exact solution of functional equation (2) will be obtained. For example, let us come back to the given Eq. (2.1) and solve it via matrix representation (3.1).

By replacing the values $S_0(f)=f(0),\ \Phi_0(x)=1,\ S_1(f)=f'(2),\ \Phi_1(x)=x^2, S_2(f)=\int_0^1 f(x)\mathrm{d}x$ and $\Phi_2(x)=x^3$ in (3.1) one gets

$$\begin{bmatrix} 0 & 0 & 0\\ 0 & 3 & 12\\ 1 & 1/3 & -3/4\end{bmatrix}\begin{bmatrix} f(0)\\ f'(2)\\ \int_0^1 f(x)\mathrm{d}x\end{bmatrix}=0. \tag{4}$$

If for instance $f(0)\neq 0$ in (4), then the solution of functional equation (2.1) would be $f(x)=f(0)(1-\frac{48}{25}x^2+\frac{12}{25}x^3)$.

But the next question is: What will happen if the second given example in (2.2) is assumed to be a functional equation?

To answer this question, it is clear that we should first constitute the elements of matrix M corresponding to Eq. (2.2). Hence, if the values $S_0(f)=f(0),\ \Phi_0(x)=1,\ S_1(f)=f'(0), \Phi_1(x)=x, S_2(f)=f''(0), \Phi_2(x)=x^2/2, S_3(f)=f'''(0)$ and $\Phi_3(x)=x^3/6$ are replaced in (3.1) then the following system of equations will be revealed

$$\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} f(0)\\ f'(0)\\ f''(0)\\ f'''(0)\end{bmatrix}=0. \tag{5}$$

Relation (5) shows that the linear system (3) (or equivalently (3.1)) always holds for the example (2.2) if $f^{(i)}(0)\neq\infty;\ i=0,1,2,3$. This means that if all elements of matrix M are zero, then the functional equation corresponding to (2) will be transformed to a finite approximation for $f(x)$. Clearly this result depends on making some suitable conditions under which the elements of matrix M are all zero. So, by looking at matrix M in (3.1), we directly find out that the fundamental condition

$$S_j\left(\Phi_i(x)\right)=\delta_{i,j}=\begin{cases}0 & \text{if } i\neq j,\\ 1 & \text{if } i=j,\end{cases} \tag{6}$$