Journal of Computational and Applied Mathematics 234 (2010) 488–504

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Exponentially small expansions in the asymptotics of the Wright function

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ARTICLE INFO

Article history:
Received 17 September 2009
Received in revised form 21 December 2009

MSC:
33C20
33C70
34E05
41A60

Keywords:
Asymptotics
Exponentially small expansions
Wright function
Generalised hypergeometric functions

ABSTRACT

We consider exponentially small expansions present in the asymptotics of the generalised hypergeometric function, or Wright function, ${}_p\Psi_q(z)$ for large $|z|$ that have not been considered in the existing theory. Our interest is principally with those functions of this class that possess either a finite algebraic expansion or no such expansion and with parameter values that produce exponentially small expansions in the neighbourhood of the negative real z axis. Numerical examples are presented to demonstrate the presence of these exponentially small expansions.

1. Introduction

We consider the generalised hypergeometric function, or Wright function, defined by

$$ {}_p\Psi_q(z) = \sum_{n=0}^{\infty} g(n)\frac{z^n}{n!}, \quad g(n) = \frac{\prod_{r=1}^{p} \Gamma(\alpha_r n + a_r)}{\prod_{r=1}^{q} \Gamma(\beta_r n + b_r)}, \tag{1.1} $$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. We also assume that the α_r and a_r are subject to the restriction

$$ \alpha_r n + a_r \neq 0, -1, -2, \ldots \quad (n = 0, 1, 2, \ldots; 1 \leq r \leq p) \tag{1.2} $$

so that no gamma function in the numerator in (1.1) is singular. In the special case $\alpha_r = \beta_r = 1$, the function ${}_p\Psi_q(z)$ reduces to a multiple of the ordinary hypergeometric function ${}_pF_q((a)_p; (b)_q; z)$ [1, p. 40].

We introduce the parameters associated with $g(n)$ given by

$$ \kappa = 1 + \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r, \qquad h = \prod_{r=1}^{p} \alpha_r^{\alpha_r} \prod_{r=1}^{q} \beta_r^{-\beta_r}, $$

$$ \vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}(q-p), \qquad \vartheta' = 1 - \vartheta. \tag{1.3} $$

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doi:10.1016/j.cam.2009.12.040

If it is supposed that α_r and β_r are such that $\kappa > 0$ then ${}_p\Psi_q(z)$ is uniformly and absolutely convergent for all finite z. If $\kappa = 0$, the sum in (1.1) has a finite radius of convergence equal to h^{-1}, whereas for $\kappa < 0$ the sum is divergent for all nonzero values of z. The parameter κ will be found to play a critical role in the asymptotic theory of ${}_p\Psi_q(z)$ by determining the sectors in the z plane in which its behaviour is either exponentially large, algebraic or exponentially small in character as $|z| \to \infty$.

The determination of the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$ and finite values of the parameters has a long history; for details, see [2, Section 2.3]. The earliest asymptotic result concerning (1.1) appears to be due to Stokes [3], who used a discrete analogue of Laplace's method for integrals when $\alpha_r = \beta_r = 1$ and positive values of a_r and b_r, to obtain the leading behaviour of ${}_p\Psi_q(z)$ when $z \to +\infty$. More precise investigations of (1.1) were carried out in [4,5] and in a long and detailed investigation in [6] into the asymptotics of a more general class of integral function than (1.1). An account of the derivation of the asymptotic expansion of ${}_p\Psi_q(z)$ for large $|z|$ based on the Euler–Maclaurin summation formula, together with an application of this theory to the asymptotics of the solutions of a class of high-order ordinary differential equation, is described in [2]. A discussion of the properties of ${}_0\Psi_1(z)$ (the generalised Bessel function) and its application to the solution of fractional diffusion-wave equations has been given in [7].

The development of exponentially precise asymptotics during the past two decades has shown that retention of exponentially small expansions, which had previously been neglected in asymptotics, is vital for a high-precision description; see the review papers [8,9] and also [10, Section 6.3]. An earlier example, which illustrated the advantage of retaining terms that are exponentially small compared with other terms in the asymptotic expansion of a certain integral, was given in [11, p. 76]. Although such terms are negligible in the Poincaré sense, their inclusion can significantly improve the numerical accuracy. In this paper we shall be concerned with exponentially small contributions present in the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$. Such terms are of particular significance when the parameters in (1.1) are such that there is a sector enclosing the negative real axis in which the dominant asymptotic behaviour of ${}_p\Psi_q(z)$ is either exponentially small or involves a finite algebraic expansion. Numerical examples are given to demonstrate the presence of these exponentially small subdominant contributions.

The paper is structured as follows. In Section 2 we present a summary of the standard results concerning the asymptotics of ${}_p\Psi_q(z)$ for large $|z|$. In Section 3 we consider exponentially small expansions present in ${}_0\Psi_q(z)$, which possess no algebraic expansion, and in Section 4 describe numerical calculations that demonstrate their existence. In Section 5 we discuss the case when ${}_p\Psi_q(z)$ possesses an algebraic expansion consisting of a finite number of terms. An algorithm for the computation of the coefficients appearing in the exponential expansions of ${}_p\Psi_q(z)$ is given in an appendix.

2. Standard asymptotic theory for $|z| \to \infty$

In this section we state the standard asymptotic expansions of the integral function ${}_p\Psi_q(z)$ as $|z| \to \infty$ with $\kappa > 0$ and finite values of the parameters given in [5,6]; see also [10, Section 2.3]. To present these results we first introduce the exponential expansion $E(z)$ and the algebraic expansion $H(z)$ associated with ${}_p\Psi_q(z)$, together with an integral representation that will be used in our discussion.

2.1. Preliminaries

The exponential expansion of $E(z)$ is given by the formal asymptotic sum

$$E(z) = Z^{\vartheta} e^{Z} \sum_{j=0}^{\infty} A_j Z^{-j}, \quad Z = \kappa (hz)^{1/\kappa}, \tag{2.1}$$

where the coefficients A_j are those appearing in the inverse factorial expansion of $g(s)/s!$ given by

$$\frac{g(s)}{\Gamma(s+1)} = \kappa (h\kappa^{\kappa})^{s} \left\{ \sum_{j=0}^{M-1} \frac{A_j}{\Gamma(\kappa s + \vartheta' + j)} + \frac{O(1)}{\Gamma(\kappa s + \vartheta' + M)} \right\} \tag{2.2}$$

for $|s| \to \infty$ uniformly in $|\arg s| \le \pi - \epsilon$, $\epsilon > 0$; see (A.1). The leading coefficient A_0 is specified by

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)} \kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^{p} \alpha_r^{a_r - \frac{1}{2}} \prod_{r=1}^{q} \beta_r^{\frac{1}{2} - b_r}. \tag{2.3}$$

The coefficients A_j are independent of s and depend only on the parameters p, q, α_r, β_r, a_r and b_r. An algorithm for their evaluation in specific cases is described in Appendix A.

The algebraic expansion $H(z)$ follows from the Mellin–Barnes integral representation [10, Section 2.3]

$${}_p\Psi_q(z) = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \Gamma(s) g(-s) (z e^{\mp \pi i})^{-s} ds, \qquad |\arg(-z)| < \tfrac{1}{2}\pi(2-\kappa) \tag{2.4}$$