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A framework for obtaining guaranteed error bounds for finite element approximations

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ABSTRACT

We give an overview of our recent progress in developing a framework for the derivation of fully computable guaranteed posteriori error bounds for finite element approximation including conforming, non-conforming, mixed and discontinuous finite element schemes. Whilst the details of the actual estimator are rather different for each particular scheme, there is nonetheless a common underlying structure at work in all cases. We aim to illustrate this structure by treating conforming, non-conforming and discontinuous finite element schemes in a single framework. In taking a rather general viewpoint, some of the finer details of the analysis that rely on the specific properties of each particular scheme are obscured but, in return, we hope to allow the reader to 'see the wood despite the trees'.

1. Introduction

The finite element method [1,2] is the pre-eminent method for the solution of partial differential equations arising in a wide range of application areas, particularly in engineering disciplines. By its very nature, the method produces an approximation of the true solution of the partial differential equation and, in view of the fact that safety critical decisions are based on the fidelity of the numerical approximation, it is of considerable interest to quantify the accuracy of the finite element approximation numerically. Typically, in order to obtain a realistic estimate of the accuracy of a given approximation, the finite element approximation of the problem itself is used *a posteriori* to assess the accuracy and, particularly in the context of standard finite element schemes, this process has reached a reasonable level of maturity [3–7].

Nevertheless, the finite element method itself is still the subject of active research and the limitations of the standard finite element procedure for particular classes of problem has resulted in a wide variety of differing schemes including *mixed finite elements*, *non-conforming finite elements*, and *discontinuous Galerkin finite elements* becoming popular. The topic of a posteriori error estimation for these less standard finite element schemes is considerably less well developed than for the case of standard finite elements. Moreover, the types of a posteriori error estimation techniques that have been developed for such schemes seem to be quite different from those developed for standard conforming finite elements with few, if any, common guiding principles emerging. One of the first works to address a posteriori estimation for the non-conforming finite element scheme of Crouzeix and Raviart [8] was the important paper of Dari et al. [9] who obtained two sided bounds on the error measured in the energy norm up to generic constants using a technique based on the Helmholtz decomposition. These ideas were later extended to non-conforming mixed finite element approximation of Stokes flow [10]. Hierarchical basis type estimators were explored in [11], whilst [12] derived estimators based on gradient averaging techniques. These works all

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give an a posteriori error upper bound involving an unknown, generic coefficient and as such do not provide actual numerical bounds. The situation regarding the discontinuous finite element method is even less developed. An explicit residual type estimator for the error measured in the energy norm was derived in [13,14] while Rivière and Wheeler [15] derived residual estimators for the error in the L_2-norm. Subsequently, Karakashian and Pascal [16] discussed several different estimators for a discontinuous finite element scheme. Once again, it is found that all of the estimators involve generic unknown constants in the upper bound, so that one does not obtain actual numerical bounds on the error.

The goal of the present work is to give an overview of some of our own recent progress in developing a framework for the derivation of a posteriori error estimators for finite element approximation in all its different ' flavours'. Our objective has primarily been the derivation of what we term *fully computable a posteriori error bounds*. The classical approach to a posteriori error estimation, as outlined in, e.g. [3], produces an *error estimator* η which takes the form of a real number which can be computed in terms of the finite element approximation, the data for the original partial differential equation and the sizes of the elements in the mesh. The estimator η is said to be *reliable* if it can be established that there exists a positive *constant C* which is independent of the size of the elements and the data of the problem for which the error e in the finite element approximation satisfies

$$|||e||| \leq C\eta \tag{1}$$

where $|||\cdot|||$ is an appropriate norm. The estimator is said to be *efficient* if (ignoring so-called data errors, see later), there exists a positive constant $c > 0$ such that

$$c\eta \leq |||e|||. \tag{2}$$

These properties are useful in the sense that they assert that the estimator is an equivalent measure of the error, but are limited in the sense that, in practice, the constant C is unknown. It is possible, at least in principle, to obtain bounds for the value of the constant, but even in the cases where this can be realised, the resulting bounds are generally overly pessimistic to the extent that they are of little value in quantifying the accuracy of practical computations.

A fully computable a posteriori error estimator is similar in that a real number η is produced from the finite element approximation, the data for the partial differential equation and the finite element mesh, but differs in that it is required to produce a guaranteed bound on the actual error:

$$|||e||| \leq \eta. \tag{3}$$

The key point is the absence of any 'generic' constant C in the upper bound. Of course, if a bound on the C for a reliable estimator is known, then one obtains a fully computable error bound. The essential feature of a *fully computable a posteriori error estimator* is that in specifying the estimator, one must give *all information necessary to produce a guaranteed and computable upper bound*. We also implicitly require the fully computable estimators to be efficient in the sense described above, and also require the estimator to be computable using only local computations on the mesh (as opposed to, say, solving a global dual problem or approximating the same problem using, for example, a globally refined mesh).

Given such stringent requirements, it is perhaps remarkable that one can produce any fully computable error bounds at all for any type of finite element scheme. However, it turns out that this is not only possible, but can be achieved for all flavours of finite element including *non-conforming finite elements* [17–21], *discontinuous Galerkin finite element schemes* [22–24] and *mixed finite element schemes* [25]. Moreover, whilst the details of the actual estimator are rather different for each particular scheme, there is nonetheless an underlying common structure at work in all cases. One of the aims of the present work is to illustrate the key structure underlying the above approach by treating conforming, non-conforming and discontinuous finite element schemes in a single framework in contrast to the usual approach of analysing each particular scheme in detail and in isolation. Of course, in presenting such an all embracing approach it is necessary to omit some of the finer details of the analysis that rely on the specific properties of the scheme. By way of compensation for the loss in details, we hope to enable the reader to ' see the wood despite the trees'. The ideas are presented in the context of piecewise affine finite element approximation of a second-order elliptic problem but it is worth emphasising that the same principles have been found to be applicable to higher order finite element approximation in the above references. The work updates and extends our previous publication [26] in this direction. A different approach to the derivation of computable error bounds for finite element approximations can be found in [27] whilst applications to finite volume methods are given in [28].

2. Preliminaries

2.1. Model problem

Consider the following model problem

$$\left.\begin{aligned} -\,\mathbf{div}\,\boldsymbol{\sigma}(u) &= f \\ \boldsymbol{\sigma}(u) - a\,\mathbf{grad}\,u &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{4}$$

subject to $u = 0$ on Γ_D and $\sigma_\nu(u) = \boldsymbol{\nu}\cdot\boldsymbol{\sigma}(u) = g$ on Γ_N, where Ω is a simple plane polygonal domain and the disjoint sets Γ_D and Γ_N form a partitioning of the boundary $\Gamma = \partial\Omega$ of the domain. Here, $\boldsymbol{\nu}$ denotes the unit, outward pointing