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Existence of multiple positive solutions of singular nonlinear boundary value problems

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ABSTRACT

For a given positive integer N, we provide conditions on the nonlinear function f which guarantee that the boundary value problem

$$y'' + f(t, y) = 0, \quad 0 < x \le r, \qquad y(0) = 0, \quad y'(r) = 0,$$

has N positive solutions. The nonlinear function f is allowed to be singular at $y = 0$ and $t = 0$ but is required to satisfy an integrability condition reminiscent of a condition first used by S. Taliaferro in 1979 and growth conditions as y increases similar to those assumed in the nonsingular case by Henderson and Thompson in 2000. Our main results depend on shooting methods.

1. Introduction

We begin with nonlinear boundary value problems of the form

$$y'' + f(t, y) = 0, \quad 0 < t < 2r, \tag{1}$$

with Dirichlet boundary conditions

$$y(0) = 0, \qquad y(2r) = 0, \tag{2}$$

where $f(t, y) \ge 0$ and $f(t, y)$ is continuous for $y > 0$, $0 < t < 2r$. We are interested in the singular case where $f(t, y) \to \infty$ as $y \to 0$, $t \to 0^+$, or $t \to 2r^-$. We shall formulate conditions which guarantee the existence of multiple positive solutions, but our results are not as definitive as we would like. Our goal is to combine two distinct types of results; one type deals with multiple solutions to nonsingular problems, and the other deals with unique solutions to singular problems.

A flurry of results of the first type, e.g. [1–5] began with the paper of Henderson and Thompson [6], where $f(t, y) = f(y)$ is independent of t with no singularity in y. They used a fixed point theorem of Leggett and Williams [7] to describe conditions on f which guarantee the existence of at least three positive solutions. Henderson and Thompson [8] have also used these methods to prove results on the existence of three positive solutions to certain boundary value problems of order $2n$. We note that these results stem originally from the paper [9] concerned with a problem arising in chemical reactor processes.

For ease of reference, we begin with a statement of the theorem of Henderson and Thompson. Consider the nonlinear boundary value problem of the form

$$y'' + f(y) = 0, \quad 0 \le t \le 1, \tag{3}$$

$$y(0) = 0, \qquad y(1) = 0. \tag{4}$$

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Theorem 1.1. *Suppose that* $0 < a < b < c/2$ *and* $f : \mathbb{R} \to [0, \infty)$ *is continuous and satisfies*

$f(y) < 8a$ *for* $0 \le y \le a$,
$f(y) \ge 16b$ *for* $b \le y \le 2b$,
$f(y) \le 8c$ *for* $0 \le y \le c$.

Then the boundary value problem (3) *and* (4) *has three symmetric positive solutions* y_1, y_2, y_3 *with* $\max y_1(x) < a$, $y_2(1/2) > a$, $y_2(1/4) < b$, $y_3(1/4) > b$, *and* $\max y_3(x) < c$.

A few comments should be made.

(1) With the hypotheses as stated, then $f(0) = 0$ is possible and the solution y_1 could be trivial, so the word positive was used here in the sense of nonnegative.

(2) The lower bound ($f(y) \ge 16b$) only needs to hold on the interval $b \le y \le 3b/2$.

(3) If the underlying interval is $0 < t < 2$ or the boundary condition at $t = 1$ is changed to $y'(1) = 0$, then the 8 and 16 are replaced by 2 and 4 respectively.

(4) This theorem has been extended and proved using at least 3 different approaches by persons in papers too numerous to catalog here.

We now shift our attention to the second type of result, in particular to work done on singular problems in [10]. Consider the second order nonlinear singular boundary value problem of the form

$$y'' + \phi(t)y^{-\lambda} = 0, \quad 0 \le t \le 1, \ \lambda > 0, \tag{5}$$

$$y(0) = y(1) = 0. \tag{6}$$

Theorem 1.2. *The boundary value problem* (5) *and* (6) *has a positive solution if and only if*

$$\int_0^1 t(1-t)\phi(t)\mathrm{d}t < \infty.$$

In this case the solution is unique.

We make some further remarks.

(1) If $y(1) = 0$ is replaced by $y'(1) = 0$, the theorem remains true as stated, except that the $(1-t)$ factor is omitted from the integrability condition.

(2) Note the very special character of the nonlinearity $\phi(t)y^{-\lambda}$.

The theorem has been extended to much more general nonlinearities $f(t, y)$ in [11–13]. We emphasize that these papers all required that $f(t, y)$ be decreasing in y for fixed t.

The contrast between these types of results is clear. The results on multiple solutions identify the cause of the multiplicity as residing in the tendency of the nonlinear function $f(y)$ to increase; whereas, in the Taliaferro theorem, the function $f(t, y) = \phi(t)y^{-\lambda}$ decreases in y. We combine these contrasting elements below by requiring that our nonlinear function $f(t, y)$ be unbounded near $y = 0$, but exhibit the Henderson–Thompson behavior for larger values of y.

Consider the general problem (1) and (2) in the case that $f(t, y)$ is symmetric in t about $t = r$. In this case, any solution of (1) on $[0, r]$ with $y(0) = 0, y'(r) = 0$ can be reflected across $t = r$ to give a solution of (1) and (2). Thus we shall focus on the boundary value problem (BVP)

$$y'' + f(t, y) = 0, \quad 0 < t \le r, \tag{7}$$

$$y(0) = 0, \qquad y'(r) = 0. \tag{8}$$

Before attacking the singular problem, it is helpful to discuss nonsingular problems which differ from the ones studied by Henderson–Thompson in two ways: (1) we allow $f(t, y)$ to depend on t; and (2) we allow $f(t, y)$ to be large, although not unbounded, near 0.

Until further notice we take $r = 1$; other values of $r > 0$ will be considered later. We state a variety of hypotheses on the nonlinear function f in (7) and (8) which will be needed throughout this work; different subsets of these hypotheses will be required for different results. We let α, a, b, c, Q denote positive parameters. Shooting methods will be heavily used and depend critically on continuous dependence of solutions of initial value problems on initial conditions. The Lipschitz condition in the first hypothesis below permits these methods.

$H_1(\alpha, c, Q)$: $f : [0, 1] \times [0, \alpha + c] \to [0, Q]$ is continuous and $f(t, y)$ satisfies a uniform Lipschitz condition in y on $[0, 1] \times [0, \alpha + c]$, i.e. there exists a constant $K > 0$ so that $|f(t, y_2) - f(t, y_1)| \le K|y_2 - y_1|$ whenever $y_1, y_2 \in [0, \alpha + c]$
$H_2(\alpha, c)$: $f(t, y) \le 2c$, $0 \le t \le 1$, $\alpha \le y \le \alpha + c$
$H_2'(\alpha, c)$: $f(t, y) < 2c$, $0 \le t \le 1$, $\alpha \le y \le \alpha + c$
$H_3(b)$: $f(t, y) \ge 4b$, $1/2 \le t \le 1$, $b \le y \le 3b/2$
$H_3'(b)$: $f(t, y) > 4b$, $1/2 \le t \le 1$, $b \le y \le 3b/2$
$H_4(\alpha, c)$: $f : ([0, 1] \times [\alpha, \alpha + c]) \cup ((0, 1] \times (0, \alpha)) \to [0, \infty)$ is continuous
$H_5(\alpha, c)$: There exist arbitrarily small values of $\eta > 0$ so that $f(t, \eta) \ge f(t, y)$, $\eta \le y \le c + \alpha$ and $\int_0^1 tf(t, \eta)\mathrm{d}t < \infty$.