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General linear methods for Volterra integral equations

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ABSTRACT

We investigate the class of general linear methods of order p and stage order $q = p$ for the numerical solution of Volterra integral equations of the second kind. Construction of highly stable methods based on the Schur criterion is described and examples of methods of order one and two which have good stability properties with respect to the basic test equation and the convolution one are given.

1. Introduction

In this paper we investigate general linear methods (GLMs) for the numerical solution of Volterra integral equations of the second kind

$$y(t) = g(t) + \int_{t_0}^{t} k\big(t, \tau, y(\tau)\big)\mathrm{d}\tau, \quad t \in [t_0, T]. \tag{1.1}$$

Here, the functions $g(t)$, $k(t, \tau, y)$ are assumed to be sufficiently smooth to guarantee the existence and the uniqueness of the solution (see [1]). The behavior of solutions to (1.1) and some of its special cases is discussed in [1–3]. Let $t_n = t_0 + nh$, $n = 0, 1, \ldots, N$, $Nh = T - t_0$, be a given uniform grid. Then to formulate numerical methods for (1.1) it is convenient to rewrite this equation for $t \in [t_n, t_{n+1}]$ in the form

$$y(t) = F^{[n]}(t) + \Phi^{[n+1]}(t) \tag{1.2}$$

with the lag term $F^{[n]}(t)$ defined by

$$F^{[n]}(t) = g(t) + \int_{t_0}^{t_n} k(t, \tau, y(\tau))\mathrm{d}\tau$$

and the increment term $\Phi^{[n+1]}(t)$ defined by

$$\Phi^{[n+1]}(t) = \int_{t_n}^{t} k(t, \tau, y(\tau))\mathrm{d}\tau,$$

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where we have suppressed the dependence of $F^{[n]}(t)$ and $\Phi^{[n+1]}(t)$ on $y(t)$. We consider the class of GLMs for (1.2) defined by

$$\begin{cases} Y_i^{[n+1]} = \sum_{j=1}^{s} a_{ij}\Big(F_h^{[n]}(t_{n,j}) + \Phi_h^{[n+1]}(t_{n,j})\Big) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, & i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = \sum_{j=1}^{s} b_{ij}\Big(F_h^{[n]}(t_{n,j}) + \Phi_h^{[n+1]}(t_{n,j})\Big) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, & i = 1, 2, \ldots, r, \end{cases} \tag{1.3}$$

$n = 1, 2, \ldots, N-1$, where s is the number of internal stages and r is the number of external stages which propagate from step to step, $t_{n,j} = t_n + c_j h$, $j = 1, 2, \ldots, s$, and $F_h^{[n]}(t_{n,j})$, $\Phi_h^{[n+1]}(t_{n,j})$ are approximations of sufficiently high order to $F^{[n]}(t_{n,j})$, $\Phi^{[n+1]}(t_{n,j})$. These approximations depend on Y_l^{ν} and $Y_l^{[n+1]}$, compare formulas (1.4) and (1.5). The methods (1.3) are modelled on GLMs for ordinary differential equations, compare [4,5,16]. Similar formulation of GLMs for Volterra integro-differential equations with delay terms was proposed in [6]. We refer to the monograph [1] for the overview of classical linear multistep methods and Runge–Kutta methods for Volterra integral and integro-differential equations including equations with weakly singular kernels.

It is assumed here (similarly to in [7,8]) that

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

and, assuming the solution has derivatives up to order r, that $y^{[n]}$ approximates the Nordsieck vector $z(t, h)$ defined by

$$z(t,h) = \begin{bmatrix} y(t) \\ \frac{h}{1!} y'(t) \\ \vdots \\ \frac{h^{r-1}}{(r-1)!} y^{(r-1)}(t) \end{bmatrix}$$

at the point $t = t_n$, i.e.,

$$y_i^{[n]} = \frac{h^{i-1}}{(i-1)!} y^{(i-1)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r.$$

The integer q is called the stage order and p the order of the method (1.3). It is also assumed that the approximations $F_h^{[n]}(t_{n,j})$ and $\Phi_h^{[n+1]}(t_{n,j})$ to $F^{[n]}(t_{n,j})$ and $\Phi^{[n+1]}(t_{n,j})$ take the following forms

$$F_h^{[n]}(t_{n,j}) = g(t_{n,j}) + h \sum_{\nu=1}^{n} \sum_{l=1}^{s} b_l k(t_{n,j}, t_{\nu-1,l}, Y_l^{[\nu]}), \tag{1.4}$$

and

$$\Phi_h^{[n+1]}(t_{n,j}) = h \sum_{l=1}^{s} w_{j,l} k(t_{n,j}, t_{n,l}, Y_l^{[n+1]}), \tag{1.5}$$

with the weights b_l and w_{jl} precomputed in advance. Observe that these formulas employ only the values of the function k at the points $t_{\nu,l}$ and not at the gridpoints t_ν.

Introducing the notation

$$Y^{[n+1]} = \begin{bmatrix} Y_1^{[n+1]} \\ \vdots \\ Y_s^{[n+1]} \end{bmatrix}, \qquad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}, \qquad F_h^{[n+1]}(t_{n,c}) = \begin{bmatrix} F_h^{[n+1]}(t_{n,1}) \\ \vdots \\ F_h^{[n+1]}(t_{n,s}) \end{bmatrix}, \qquad \Phi_h^{[n]}(t_{n,c}) = \begin{bmatrix} \Phi_h^{[n]}(t_{n,1}) \\ \vdots \\ \Phi_h^{[n]}(t_{n,s}) \end{bmatrix},$$

and

$$A = \big[a_{ij}\big] \in R^{s\times s}, \qquad U = \big[u_{ij}\big] \in R^{s\times r}, \qquad B = \big[b_{ij}\big] \in R^{r\times s}, \qquad V = \big[v_{ij}\big] \in R^{r\times r},$$

the method (1.3) can be rewritten in a more compact vector form

$$\begin{cases} Y^{[n+1]} = A\Big(F_h^{[n]}(t_{n,c}) + \Phi_h^{[n+1]}(t_{n,c})\Big) + Uy^{[n]}, \\ y^{[n+1]} = B\Big(F_h^{[n]}(t_{n,c}) + \Phi_h^{[n+1]}(t_{n,c})\Big) + Vy^{[n]}, \end{cases} \tag{1.6}$$

$n = 0, 1, \ldots, N-1$. This form will be convenient to analyze stability properties of (1.3) with respect to the basic and the convolution test equations.