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Fast Fourier–Galerkin methods for solving singular boundary integral equations: Numerical integration and precondition

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ABSTRACT

We develop a fast fully discrete Fourier–Galerkin method for solving a class of singular boundary integral equations. We prove that the number of multiplications used in generating the compressed matrix is $\mathcal{O}(n \log^3 n)$, and the solution of the proposed method preserves the optimal convergence order $\mathcal{O}(n^{-t})$, where n is the order of the Fourier basis functions used in the method and t denotes the degree of regularity of the exact solution. Moreover, we propose a preconditioning which ensures the numerical stability when solving the preconditioned linear system. Numerical examples are presented to confirm the theoretical estimates and to demonstrate the approximation accuracy and computational efficiency of the proposed algorithm.

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1. Introduction

Fourier–Galerkin methods are widely used in solving boundary integral equations [1–12]. In [4], a fast algorithm was proposed, whose error analysis was presented in [13]. A fully discrete Galerkin boundary element method with linear complexity was introduced in [2]. In [3], a fast Fourier–Galerkin method is used for solving singular boundary integral equations, where a compression strategy is applied to the coefficient matrices generated by the method. This fast Fourier–Galerkin method solves a sparse linear system with a coefficient matrix having only $\mathcal{O}(n \log n)$ nonzero entries, and enjoys the optimal convergence order $\mathcal{O}(n^{-t})$, where n denotes the order of the Fourier basis functions used in this method and t the degree of regularity of the exact solution. This paper continues the theme of [3] to develop an efficient numerical quadrature scheme for the fast Fourier–Galerkin method which preserves the optimal convergence order of the original method, and uses only *quasi-linear* numbers of multiplications for computing all nonzero entries. Specifically, we prove that the number of multiplications used in generating the coefficient matrix is $\mathcal{O}(n \log^3 n)$, and the corresponding approximate solution has the optimal convergence order $\mathcal{O}(n^{-t})$.

We face a challenging issue of developing a numerical integration method to efficiently compute nonzero coefficients. When the frequencies of Fourier basis functions are much greater than 1, computing the corresponding coefficients involves two-dimensional oscillatory integrals. We require that computing all nonzero entries uses *only* quasi-linear number of multiplications. To treat these oscillatory integrals, we adopt the product integration method which was originated in [14] so that the integrals of the oscillatory factors are evaluated exactly. With the product integration method, we can obtain

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sufficient precision to ensure that the solution has the optimal convergence order. For recent development of numerical integration of oscillatory integrals, see [15–19]. The key idea to achieve the computational complexity requirement is to use a quadrature strategy developed in [20]. Namely, we construct an approximation of the non-oscillatory factor of the integrand with no more than $\mathcal{O}(n \log n)$ number of functional evaluations for computing all integrals needed for the sparse Fourier expansion. We then write the resulting numerical quadrature formula as discrete Fourier transforms of vectors of the coefficients of Lagrange piecewise polynomial basis functions so that the fast Fourier transform [21] can be applied. To construct the approximation of the non-oscillatory factor of the integrand, we employ the multiscale Lagrange interpolation on sparse grids by using the mathematical development presented in [20,22,23]. The recent research progress of sparse grid methods can be found in a mastery review paper [24] and other publications [25–32]. Note that multidimensional integrals were treated in [33] in the context of expanding a given function by the spherical harmonic functions. In a more general context, lattice rules were proposed in [34,35] to efficiently evaluate multidimensional integrals.

This paper is organized in six sections with an Appendix. In Section 2, we review the truncation strategy [3] for solving singular boundary integral equations and develop its fully discrete scheme. We also estimate the computational complexity of the proposed algorithm. This is, we prove that by the proposed fully discrete method, the number of multiplications used in generating the coefficient matrix is $\mathcal{O}(n \log^3 n)$, where n is the order of the Fourier basis functions used in the method. We prove in Section 3 that the proposed method is stable and achieves the optimal convergence order. In Section 4, we introduce a precondition for the compressed coefficient matrix that ensures the condition number of the preconditioned matrix is bounded by a constant independent of the matrix size. We present in Section 5 numerical examples to confirm the approximation accuracy and computational efficiency of the proposed method and to demonstrate the use of the method in solving boundary value problem of the Laplace equation. We draw a conclusion in Section 6. For convenience of the readers, we also add an Appendix at the end of this paper to review the fast quadrature scheme crucial for the development of the main algorithm described in Section 2.

2. A fast fully discrete Fourier–Galerkin method

In this section, we review the fast Fourier–Galerkin method introduced in [3] for solving singular boundary integral equations and develop its fully discrete scheme by using a numerical integration method introduced in [20].

Let $I := [0, 2\pi]$ and $\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$. For $k \in \mathbb{Z}$, we set $e_k(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}kx}$, for $x \in I$, where i is the imaginary unit. For each $s \geq 0$, we denote by $H^s(I)$ the standard Sobolev space which consists of functions $\phi \in L^2(I)$ with property $\sum_{k\in\mathbb{Z}}(1+k^2)^s|\phi_k|^2 < +\infty$, where $\phi_k := \langle \phi, e_k \rangle$ are the Fourier coefficients of ϕ. The inner product of space $H^s(I)$ is defined for $\phi, \psi \in H^s(I)$ by $\langle \phi, \psi \rangle_s := \sum_{k\in\mathbb{Z}}(1+k^2)^s \phi_k \bar{\psi}_k$, $\phi, \psi \in H^s(I)$, and its norm is defined by $\|\phi\|_s := \langle \phi, \phi \rangle_s^{\frac{1}{2}}$. For two nonnegative numbers s_1 and s_2, we suppose that $X \subseteq H^{s_1}(I)$ and $Y \subseteq H^{s_2}(I)$. Let $\mathcal{A} : X \to Y$ be a bounded linear operator defined by

$$(\mathcal{A}w)(x) = \int_I a(x, y) w(y) \mathrm{d}y, \quad x \in I,$$

which has a bounded inversion $\mathcal{A}^{-1} : Y \to X$. Throughout this paper, we always assume that operator $\mathcal{A}$ has the Fourier basis functions e_k as its eigenfunctions. The kernel a of operator $\mathcal{A}$ may be either weakly singular, strong singular, or hypersingular. We assume that $\mathcal{B} : X \to Y$ is a compact operator defined by

$$(\mathcal{B}w)(x) = \int_I b(x, y) w(y) \mathrm{d}y, \quad x \in I,$$

where b is a smooth kernel. The boundary integral equations considered in this paper have the form

$$(\mathcal{A} + \mathcal{B}) u = g, \tag{2.1}$$

where $g \in Y$ is a given function and $u \in X$ is the unknown to be determined. Many boundary integral equations can be written in the form (2.1). For example, both the interior and exterior Dirichlet problems in the planes with smooth boundaries have the form (2.1) with the kernel a being weakly singular. The interior Neumann problem can be rewritten in the form (2.1) with the strong singular or hypersingular kernel a. Deviations of the boundary integral equations for the boundary value problems mentioned above can be found in [1]. More discussion on reformulation of boundary value problem of the Laplace equation will be given in Section 5.

We now review the Fourier–Galerkin method for solving the boundary integral equations (2.1). Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_n^+ := \{1, 2, \ldots, n-1\}$ and $\mathbb{Z}_n := \mathbb{Z}_n^+ \cup \{0\}$. For each $n \in \mathbb{N}$, we define a finite-dimensional subspace X_n by $X_n := \mathrm{span}\{e_k : |k| \in \mathbb{Z}_n^*\}$, where $\mathbb{Z}_n^*$ denotes either $\mathbb{Z}_n$ or $\mathbb{Z}_n^+$, according to the type of integral operators $\mathcal{A}$ in Eq. (2.1). Let P_n denote the orthogonal projection from X to X_n. The Fourier–Galerkin method for solving Eq. (2.1) is to seek $u_n \in X_n$ such that

$$(P_n\mathcal{A} + P_n\mathcal{B}) u_n = P_n g. \tag{2.2}$$

Since operator $\mathcal{A}$ has the Fourier basis functions e_k as its eigenfunctions, operator $\mathcal{A}$ commutes with the projection P_n, that is

$$\mathcal{A}P_n = P_n\mathcal{A}. \tag{2.3}$$

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