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From the Ehrenfest model to time-fractional stochastic processes

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ABSTRACT

The Ehrenfest model is considered as a good example of a Markov chain. I prove in this paper that the time-fractional diffusion process with drift towards the origin, is a natural generalization of the modified Ehrenfest model. The corresponding equation of evolution is a linear partial pseudo-differential equation with fractional derivatives in time, the orders lying between 0 and 1. I focus on finding a precise explicit analytical solution to this equation depending on the interval of the time. The stationary solution of this model is also analytically and numerically calculated. Then I prove that the difference between the discrete approximate solution at time t_n, $\forall n \geq 0$, and the stationary solution obeys a power law with exponent between 0 and 1. The reversibility property is discussed for the Ehrenfest model and its fractional version with a new observation.

1. Introduction

The Ehrenfest urn model treats a wide class of stochastic processes [1,39,2]. These processes are reversible processes, i.e. when the direction of time is reversed, the behavior of the process remains the same [3]. The partial differential equation approximated by the Ehrenfest model is a special case of the Fokker Planck equation. Many methods of solution and applications of it can be found in [4]. It can also be described as a diffusion equation with a central linear drift towards the origin [5,6]. Smoluchowski [7] showed that this equation describes also the so called Ornstein–Uhlenbeck process [8]. In recent years fractional differential equations have been studied by many mathematicians, physicists and engineers, see for example [9–11], and applied in an increasing number of fields such as physics, chemistry, signal processing [12,13], control engineering, electromagnetism, fluid mechanics [14], and finance [15]. Moving from the classical Ehrenfest model to the time fractional diffusion equation with central linear drift, also called the time fractional Fokker–Planck equation (FFPE), is a nice example of passing from a discrete to a continuous model. Fractional in time means that the first-order time derivative is replaced by the Caputo derivative of order $\beta \in (0, 1]$. This generalization describes many stochastic processes [16–18]. It interprets also the subdiffusion behavior of a particle under the combined influence of external non linear field [19]. Many attempts have been made to find an explicit solution of this time-fractional partial differential equation, see for example [20–22]. The effect of fractal external force on the asymptotic behavior of the solution is also studied in [23].

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This paper is organized as follows: In Section 2, I give the analytical solution of the time-fractional partial differential equation in a new expression depending on the interval of the time. In Section 3, the discrete scheme and its relation to the Ehrenfest model is discussed. In Section 4, the stationary solution and henceforth the reversibility property will be discussed. In Section 5, the numerical results will be displayed.

2. Solution of the time-fractional diffusion equation with drift

The partial differential equation which describes the elastic diffusive motion of a bound particle (for example a small pendulum) is a special case of the Fokker–Planck equation

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + b\frac{\partial (xu(x,t))}{\partial x}, \quad a>0,\ b>0,\ -\infty<x<\infty,\ t\geq 0, \tag{2.1}$$

here a is the diffusion constant and bx is the drift term, and if $b=0$, we have the classical diffusion equation. The conditions imposed on the solution $u(x,t)$ are $u(x,t)\geq 0$ and $\int_{-\infty}^{\infty} u(x,t) = 1$. With the initial condition $u(x,0)=\delta(x-x_0)$, the solution of Eq. (2.1) reads

$$u(x,t) = p(x_0;x,t) = \frac{1}{\sqrt{2\pi\frac{a}{b}(1-\mathrm{e}^{-2bt})}}\mathrm{e}^{\frac{-(x-x_0\mathrm{e}^{-bt})^2}{\frac{a}{b}(1-\mathrm{e}^{-2bt})}},$$

see [2,24]. This stochastic process described by Eq. (2.1) is a *Markov process*. Now I will replace the time derivative $\partial/\partial t$ by the Riemann–Liouville fractional derivative operator of order β, where $0<\beta<1$ see [25,10,26], $\underset{t}{D}^{\beta}$, and get

$$\underset{t}{D}^{\beta} u(x,t) = K_\beta L_{fp} u(x,t) + \frac{t^{-\beta}\delta(x)}{\Gamma(1-\beta)}, \tag{2.2}$$

where K_β is the diffusion constant. If $\beta=1$, one gets Eq. (2.1). It is worth saying that Eq. (2.2) is a special kind of the time-fractional Fokker–Planck equation *FFPE*, see [21,17,22], and L_{fp} is called the Fokker–Plank operator, see [4],

$$L_{fp}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial F(x)u(x,t)}{\partial x}. \tag{2.3}$$

Here, $F(x)$ must be an attractive linear force [21,17,22]. One can use the alternative time-fractional derivative, see [11,10], namely the Caputo time fractional derivative which is related to Riemann–Liouville by the relation

$$\underset{t*}{D}^{\beta} = \underset{t}{D}^{\beta} - \frac{t^{-\beta}\delta(x)}{\Gamma(1-\beta)}, \quad 0<\beta<1.$$

Then Eq. (2.2) is rewritten by using the Caputo fractional derivative operator of order β as

$$\underset{t*}{D}^{\beta} u(x,t) = K_\beta L_{fp} u(x,t). \tag{2.4}$$

Another version of Eq. (2.4) can be found at [27], which I call the *time-fractional diffusion equation with central linear drift towards the origin*, and is also called the *time-fractional Uhlenbeck–Ornstein process*, see [8,28], if one use the boundary conditions $u(\pm\infty,t)=0$. Clearly, the solution of Eq. (2.2) is also a solution to Eq. (2.4) and all its versions, if one uses the same boundary conditions and the same choice of $F(x)$. Therefore, let $F(x) = -\frac{ax}{K_\beta}$, and solve Eq. (2.2) by using the method of separation of variables for which I define the new independent variables

$$\tilde{x} = \sqrt{\frac{a}{K_\beta}}x, \qquad \tilde{t} = a^{-(\beta+1)}t,$$

then, write $u(\tilde{x},\tilde{t}) = X(\tilde{x})T(\tilde{t})$. Now, after using the method of separation of variables, one gets

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tilde{x}^2} + \tilde{x}\frac{\mathrm{d}X}{\mathrm{d}\tilde{x}} + (n+1)X = 0, \tag{2.5}$$

and

$$\frac{\partial^\beta T}{\partial \tilde{t}^\beta} - \frac{\tilde{t}^{-\beta}}{\Gamma(1-\beta)} + nT = 0. \tag{2.6}$$

The solution of Eq. (2.6) is the Mittag–Leffler function, see [29,30],

$$T(\tilde{t}) = E_\beta(-n\tilde{t}^\beta) = \sum_{k=0}^{\infty}\frac{(-n)^k\tilde{t}^{\beta k}}{\Gamma(\beta k+1)}. \tag{2.7}$$