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Polynomial interior-point algorithms for $\mathrm{P}_*(\kappa)$ horizontal linear complementarity problem

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ABSTRACT

In this paper a class of polynomial interior-point algorithms for $\mathrm{P}_*(\kappa)$ horizontal linear complementarity problem based on a new parametric kernel function, with parameters $p \in [0, 1]$ and $\sigma \geq 1$, are presented. The proposed parametric kernel function is not exponentially convex and also not strongly convex like the usual kernel functions, and has a finite value at the boundary of the feasible region. It is used both for determining the search directions and for measuring the distance between the given iterate and the μ-center for the algorithm. The currently best known iteration bounds for the algorithm with large- and small-update methods are derived, namely, $O((1+2\kappa)\sqrt{n}\log n \log \frac{n}{\varepsilon})$ and $O((1+2\kappa)\sqrt{n}\log \frac{n}{\varepsilon})$, respectively, which reduce the gap between the practical behavior of the algorithms and their theoretical performance results. Numerical tests demonstrate the behavior of the algorithms for different results of the parameters p, σ and θ.

1. Introduction

Given $M \in \mathbf{R}^{n\times n}$ and $q \in \mathbf{R}^n$, the standard linear complementarity problem (SLCP) is to find a pair $x, s \in \mathbf{R}^n$ such that

$$s = Mx + q, \quad x^{\mathrm{T}}s = 0, \quad (x, s) \geq 0. \tag{1}$$

If M is positive semidefinite matrix, i.e.,

$$x^{\mathrm{T}}Mx \geq 0, \quad \forall x \in \mathbf{R}^n, \tag{2}$$

then the SLCP is called a monotone SLCP, which is a fundamental decision and optimization problem. It also arises from economic equilibrium problems, noncooperative games, traffic assignment problems, and optimization problems. For an overview of these results we refer to [1–4].

The generalization of this problem is called the horizontal linear complementarity problem (HLCP), which seeks vectors $x, s \in \mathbf{R}^n$ such that

$$Mx + Ns = q, \quad x^{\mathrm{T}}s = 0, \quad (x, s) \geq 0, \tag{3}$$

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where $q \in \mathbf{R}^n$, and $M, N \in \mathbf{R}^{n\times n}$. If M and N have the column monotonicity property, i.e.,

$$Mx + Ns = 0 \Longrightarrow x^{\mathrm{T}}s \geq 0, \tag{4}$$

then the HLCP is called a monotone HLCP.

It is worth noting that HLCP becomes the SLCP if N is nonsingular, then HLCP reduces to SLCP. HLCP also includes the standard linear optimization (LO) and convex quadratic optimization (CQO) [5]. There are a variety of solution approaches for HLCP which have been studied intensively. Among them, the interior-point methods (IPMs) gained much attention than other methods. After the seminal work of Karmarkar, many researchers have proposed IPMs for LO CQO and SLCP and achieved plentiful results [6–12]. Theoretically, HLCP can be solved by using any algorithm for SLCP [13], but directly solving HLCP is a better choice than using any algorithm for SLCP for solving the HLCP. Due to the close connection between HLCPLOCQO and SLCP, some IPMs for LO CQO and SLCP have been extended to HLCP. For instance, Gonzaga et al. [14,15] studied the largest step path following algorithm for monotone HLCP and showed that the fast convergence of the simplified largest step path following algorithm. Huang et al. [16] proposed a high-order feasible interior-point method for HLCP with $O(\sqrt{n}\log\frac{\varepsilon_0}{\varepsilon})$ iterations. Monteiro et al. [17] studied the limiting behavior of the derivatives of certain trajectories associated with the monotone HLCP. Zhang [5] presented a class of infeasible IPMs for HLCP and showed that the algorithm has $O(n^2\log\frac{1}{\varepsilon})$ under some mild assumptions. Some other relevant references can be found in [18–22].

It should be noted that all most known polynomial various of IPMs used the so-called central path as a guideline to the optimal set, and some various of the Newton method to follow the central path approximately. However there is still a gap between the practical behavior of these algorithms and these theoretical performance results with respect to the update strategies of the duality gap parameter in the algorithm. The so-called large-update IPMs have superior practical performance but with relatively weak theoretical results. While the so-called small-update IPMs enjoy the best known worst-case iteration bound but their performance in computational practice is poor. This gap was reduced by Peng et al. [10] who introduced the so-called self-regular barrier functions based on IPMs for LO and semidefinite optimization (SDO). Bai et al. [7,8,23] who presented IPMs based on a new class of non-self-regular kernel functions for LO and second-order cone optimization (SOCO) and also obtained the same best known iteration bounds for the algorithms with large- and small-update methods as they are in [10]. Recently, Bai et al. [24], Cho et al. [25] proposed large-update interior-point algorithms for $\mathrm{P}_*(\kappa)$-SLCP, i.e., for $\kappa \geq 0$, $x \in \mathbf{R}^n$, the matrix M satisfies

$$s = Mx \Longrightarrow x^{\mathrm{T}}s \geq -4\kappa\sum_{i\in I_+} x_i s_i, \qquad I_+ = I_+(x,s) = \{i : x_i s_i \geq 0\}, \tag{5}$$

based on special kernel functions, respectively. Note that for $\kappa = 0$, $\mathrm{P}_*(0)$-SLCP is the monotone SLCP.

In this paper, we consider $\mathrm{P}_*(\kappa)$-HLCP, i.e., the matrices M and N satisfy additionally the following assumption.

Assumption 1.1.

$$Mx + Ns = 0 \Longrightarrow x^{\mathrm{T}}s \geq -4\kappa\sum_{i\in I_+} x_i s_i, \quad \forall (x,s) \in \mathbf{R}^{2n}, \tag{6}$$

where κ is a nonnegative number.

It should be noted that Assumption 1.1 can guarantee that the modified Newton-system has a unique solution. The proof of the following lemma is quite similar to Lemma 4.1 in [4], so we omit it here.

Lemma 1.2. *Under Assumption 1.1, the modified Newton-system*

$$\begin{pmatrix} M\Delta x + N\Delta s \\ S\Delta x + X\Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix}$$

has a unique solution $(\Delta x, \Delta s)$ *for* $(x, s) > 0$, *where* $X = diag(x)$ *and* $S = diag(s)$.

Obviously, if $\kappa = 0$, the $\mathrm{P}_*(\kappa)$-HLCP reduces the monotone HLCP.

Kernel functions play an important role in the design and analysis for the interior-point algorithms [8,10]. They not only can be used to define the new search directions but also can be considered as a measure for the distance between the given iterate and the μ-center. In general each kernel function gives rise to an interior-point algorithm. Recently, Bai et al. [7] presented a new efficient large-update primal–dual interior-point method for LO based on a finite kernel function as follows

$$\psi_{1,\sigma}(t) = \frac{t^2-1}{2} + \frac{\mathrm{e}^{\sigma(1-t)}-1}{\sigma}, \quad \sigma \geq 1. \tag{7}$$

It is not exponentially convex and also not strongly convex like the usual kernel functions [8,10], which has a finite value at the boundary of the feasible region, i.e.,

$$\lim_{t\to 0}\psi_{1,\sigma}(t) = \psi_{1,\sigma}(0) = \frac{\mathrm{e}^{\sigma}-1}{\sigma} - \frac{1}{2} < \infty. \tag{8}$$