Journal of Computational and Applied Mathematics 233 (2009) 555–563

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Hyperasymptotics and hyperterminants: Exceptional cases

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ARTICLE INFO

Article history:
Received 20 May 2009
Received in revised form 31 July 2009

MSC:
primary 30E15
34M40
secondary 39A11

Keywords:
Asymptotic expansions
Hyperasymptotics
Recurrence relations
Stokes multiplier

ABSTRACT

A new method is introduced for the computation of hyperterminants. It is based on recurrence relations, and can also be used to compute the parameter derivatives of the hyperterminants. These parameter derivatives are needed in hyperasymptotic expansions in exceptional cases. Numerical illustrations and an application are included.

1. Hyperterminants

In the last two decades hyperasymptotic expansions were constructed for solutions of differential equations and difference equations, and for integrals with saddles. See [1–12]. In this way exponentially small phenomena were incorporated in the expansions, and it gave a powerful method to compute the so-called Stokes multipliers, or connection coefficients, to arbitrary precision [8]. Hyperasymptotic expansions also incorporate the higher-order Stokes phenomenon, which seems to play an important role in some partial differential equations, [13,6].

Hyperasymptotic expansions are in terms of hyperterminants. In [14] the hyperterminants are defined and a new integral representation is used to obtain convergent expansions for the hyperterminants in series of confluent hypergeometric functions. For the coefficients in these expansions a recursive scheme is given. These expansions can be used to compute the hyperterminants to any given accuracy.

In the papers mentioned above, the asymptotic approximations are of the form $w_j(z) \sim e^{\lambda_j z} z^{\mu_j}, j = 1, \ldots, n$, as $|z| \to \infty$. It is usually assumed that $\lambda_j \neq \lambda_k$, whenever $j \neq k$. In the case that there are j, k such that $j \neq k$, $\lambda_j = \lambda_k$ and $\mu_j - \mu_k$ is an integer, extra logarithmic factors, $\ln z$, appear in the expansions, and new methods are needed to compute the corresponding hyperterminants. For examples see [1,3] and the main application in this paper. Note that $\frac{d}{d\mu} z^\mu = \ln(z) z^\mu$. Hence, the new logarithmic factor can be seen as the result of a μ-derivative of the original expansion.

In this paper we construct an alternative method based on recurrence relations for the computation of the hyperterminants. As is shown in [15], the computation of parameter derivatives of solutions of recurrence relations is not a big problem. Taking a parameter derivative of a linear recurrence relation does not change the shape of the recurrence relation itself. Hence, if it is possible to use the recurrence relation to compute its solutions numerically, then it is also possible to use the recurrence relation to compute the parameter derivatives of its solutions.

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.08.005

The definition of the hyperterminants is

$$F^{(0)}(z) = 1$$
$$F^{(1)}\left(z; \begin{matrix} M_0 \\ \sigma_0 \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \frac{e^{\sigma_0 t_0} t_0^{M_0-1}}{z-t_0}\, dt_0 \tag{1.1}$$
$$F^{(\ell+1)}\left(z; \begin{matrix} M_0, \ldots, M_\ell \\ \sigma_0, \ldots, \sigma_\ell \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_\ell]} \frac{e^{\sigma_0 t_0 + \cdots + \sigma_\ell t_\ell} t_0^{M_0-1} \cdots t_\ell^{M_\ell-1}}{(z-t_0)(t_0-t_1)\cdots(t_{\ell-1}-t_\ell)}\, dt_\ell \cdots dt_0,$$

where we use the notation $\theta_j = \mathrm{ph}\,\sigma_j$ and $\int^{[\eta]} = \int^{\infty e^{i\eta}}$. In [14] we also give an alternative integral representation. From integral representation (1.1) it is obvious that the hyperterminants are multi-valued functions with respect to z, but also with respect to σ_j. Connection relations with respect to all these variables are given in [14].

In exceptional cases (see for example [1] and [3]) extra factors $(\ln t_j)^n$ appear in the integrand, and these new functions can be seen as parameter derivatives of the original hyperterminants:

$$\frac{\partial^n}{\partial M_j^n} F^{(\ell+1)}\left(z; \begin{matrix} M_0, \ldots, M_\ell \\ \sigma_0, \ldots, \sigma_\ell \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_\ell]} \frac{e^{\sigma_0 t_0 + \cdots + \sigma_\ell t_\ell} t_0^{M_0-1} \cdots t_\ell^{M_\ell-1} (\ln t_j)^n}{(z-t_0)(t_0-t_1)\cdots(t_{\ell-1}-t_\ell)}\, dt_\ell \cdots dt_0. \tag{1.2}$$

We could construct recurrence relations with respect to each of the M_j parameters, but in applications one mainly needs recurrence relations with respect to the final M_j parameter. In this paper we will use linear first-order recurrence relations with respect to the M_ℓ parameter. Our method requires that this parameter is not an integer. Since it is always possible to interchange the M_j parameters, the method that we give in this paper will always work, except when all the M_j are integers.

In section $\ell + 1$, $\ell = 1, 2, 3$, we discuss the computation of the level ℓ hyperterminant. Each of these sections is split in two parts: First we deal with the case that the variable $z = 0$, and then we use these results and deal with the case $z \neq 0$. In applications $|z|$ is large, but for the computation of the Stokes multipliers we will need hyperterminants with $z = 0$. In Section 3 we also give a numerical illustration.

Finally, in Section 5 we apply these results and discuss the hyperasymptotics of a linear third-order differential equation in which logarithmic factors appear.

2. Level 1

We will assume that M is not an integer and $z \neq 0$. The definition of the first hyperterminant reads

$$F^{(1)}\left(z; \begin{matrix} M \\ \sigma \end{matrix}\right) = \int_0^{[\pi-\theta]} \frac{e^{\sigma t} t^{M-1}}{z-t}\, dt$$
$$= e^{M\pi i} \sigma^{1-M} \int_0^\infty \frac{e^{-\tau} \tau^{M-1}}{z\sigma + \tau}\, d\tau = e^{M\pi i + \sigma z} z^{M-1} \Gamma(M) \Gamma(1-M, \sigma z), \tag{2.1}$$

when $|\mathrm{ph}\,(\sigma z)| < \pi$, where $\Gamma(a, z)$ is the incomplete gamma function (see Section 11.2 in [16]). The integrals in (2.1) converge for $\Re M > 0$. We use analytical continuation via the recurrence relation below to define this function for $\Re M \leqslant 0$. It follows that

$$F^{(1)}\left(0; \begin{matrix} M \\ \sigma \end{matrix}\right) = e^{M\pi i} \sigma^{1-M} \Gamma(M-1). \tag{2.2}$$

Hence,

$$\frac{\partial}{\partial M} F^{(1)}\left(0; \begin{matrix} M \\ \sigma \end{matrix}\right) = (\pi i - \ln(\sigma) + \psi(M-1))\, F^{(1)}\left(0; \begin{matrix} M \\ \sigma \end{matrix}\right), \tag{2.3}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function (see Section 3.4 in [16]).

For $r = 0, 1, 2, \ldots$, let

$$y_r = F^{(1)}\left(z; \begin{matrix} M+r \\ \sigma \end{matrix}\right) \quad \text{and} \quad y_r' = \frac{\partial y_r}{\partial M}, \tag{2.4}$$

then we have the recurrence relation

$$y_{r+1} - z y_r = F^{(1)}\left(0; \begin{matrix} M+r+1 \\ \sigma \end{matrix}\right), \tag{2.5}$$

with normalising condition

$$\sum_{r=0}^\infty \frac{(-\sigma)^r y_r}{r!} = F^{(1)}\left(z; \begin{matrix} M \\ 0\sigma \end{matrix}\right) = e^{M\pi i} z^{M-1} \Gamma(M) \Gamma(1-M) = \frac{\pi e^{M\pi i} z^{M-1}}{\sin M\pi}, \quad \Re M < 1. \tag{2.6}$$

These two results follow from the first integral representation in (2.1), where we need $0 < \Re M < 1$ for the proof of (2.6), and use analytic continuation to extend the result to $\Re M < 1$. Note that in the definitions (1.1) the phase of the