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Some sufficient conditions for the non-negativity preservation property in the discrete heat conduction model

Tamás Szabó

Eötvös Loránd University, Institute of Mathematics, 1117 Budapest, Pázmány P. S. 1/c, Hungary

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ABSTRACT

In the course of the numerical approximation of mathematical models there is often a need to solve a system of linear equations with a tridiagonal or a block-tridiagonal matrices. Usually it is efficient to solve these systems using a special algorithm (tridiagonal matrix algorithm or TDMA) which takes advantage of the structure. The main result of this work is to formulate a sufficient condition for the numerical method to preserve the non-negativity for the special algorithm for structured meshes. We show that a different condition can be obtained for such cases where there is no way to fulfill this condition. Moreover, as an example, the numerical solution of the two-dimensional heat conduction equation on a rectangular domain is investigated by applying Dirichlet boundary condition and Neumann boundary condition on different parts of the boundary of the domain. For space discretization, we apply the linear finite element method, and for time discretization, the well-known Θ-method. The theoretical results of the paper are verified by several numerical experiments.

1. Preliminaries

The preservation of characteristic qualitative properties of different phenomena is a more and more important requirement in the construction of reliable numerical models [1]. For phenomena that can be mathematically described by linear partial differential equations of parabolic type (such as heat conduction, diffusion, the pricing of options, etc.), the most important qualitative properties are: the maximum–minimum principle, the non-negativity preservation and the maximum norm contractivity [2].

By applying finite difference or finite element methods, the solution of a mathematical problem often reduces to the solution of a system of linear equations with a tridiagonal or a block-tridiagonal matrix. If the matrix $\bar{\bar{A}}$ on the left-hand side of the equation

$$\bar{\bar{A}}\bar{X} = \bar{F} \tag{1.1}$$

is a block-tridiagonal matrix with $(m+1) \times (m+1)$ blocks, then the problem is equivalent to the solution of the following system:

$$B_0X_0 - C_0X_1 = F_0, \tag{1.2}$$
$$-A_iX_{i-1} + B_iX_i - C_iX_{i+1} = F_i, \quad i = 1, \dots, m-1 \tag{1.3}$$
$$-A_mX_{m-1} + B_mX_m = F_m, \tag{1.4}$$

where the blocks $A_i, B_i, C_i \in \mathbf{R}^{n\times n}$; $X_i, F_i \in \mathbf{R}^n$.

E-mail address: szabot@cs.elte.hu.

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Usually, in the TDMA (tridiagonal matrix algorithm) [3] the solution is sought in the form

$$X_i = \alpha_{i-1} X_{i-1} + \beta_{i-1}, \quad i = 1, 2, \ldots, m \tag{1.5}$$

where $\alpha_i \in \mathbf{R}^{n\times n}$ for $i = 0, \ldots, m-1$ and $\beta_i \in \mathbf{R}^n$ for $i = -1, \ldots, m-1$. Then the solution of the system of linear algebraic equations (1.2), (1.3) and (1.4) can be defined by the following algorithm

TDMA

1. We put

$$\alpha_{m-1} = B_m^{-1} A_m, \tag{1.6}$$

and

$$\beta_{m-1} = B_m^{-1} F_m. \tag{1.7}$$

2. We find $\alpha_{m-2}, \alpha_{m-3}, \ldots, \alpha_0$ and $\beta_{m-2}, \beta_{m-3}, \ldots, \beta_{-1}$ by the formulas

$$\alpha_{i-1} = (B_i - C_i \alpha_i)^{-1} A_i, \tag{1.8}$$

and

$$\beta_{i-1} = (B_i - C_i \alpha_i)^{-1} (C_i \beta_i + F_i). \tag{1.9}$$

3. Then by the formulas

$$X_0 = \beta_{-1} \tag{1.10}$$

and (1.5) we define $X_0, X_1, \ldots, X_m$.

In the following, we will show under which conditions the matrix inverses exist in the TDMA.

Lemma 1 (*See for e.g. [4]*)**.** *Let $M \in \mathbf{R}^{n\times n}$. If $\|M\| < 1$, then $I - M$ is regular and*

$$\frac{1}{1 + \|M\|} \le \left\|(I + M)^{-1}\right\| \le \frac{1}{1 - \|M\|}. \tag{1.11}$$

Lemma 2. *Let us assume that the following conditions hold:*

$$\text{there exists } B_i^{-1}, \quad \text{for all } i = 0, 1, \ldots, m \tag{1.12}$$

$$\left\|B_0^{-1} C_0\right\| < 1, \tag{1.13}$$

$$\left\|B_i^{-1} A_i\right\| + \left\|B_i^{-1} C_i\right\| < 1, \quad i = 1, \ldots, m-1, \tag{1.14}$$

$$\left\|B_m^{-1} A_m\right\| < 1. \tag{1.15}$$

Then

$$\|\alpha_i\| < 1, \quad i = 0, \ldots, m-1. \tag{1.16}$$

Proof. For α_{m-1} the statement follows from (1.15) and (1.6). It is easy to obtain

$$\alpha_{i-1} = (I - R_i)^{-1} B_i^{-1} A_i, \tag{1.17}$$

where $R_i := B_i^{-1} C_i \alpha_i$.

We will show that $\|\alpha_i\| < 1$ implies $\|\alpha_{i-1}\| < 1$, for $i = 1, \ldots, m-1$. From (1.17) we get:

$$\|\alpha_{i-1}\| \le \left\|(I - R_i)^{-1}\right\| \left\|B_i^{-1} A_i\right\| . \tag{1.18}$$

Hence $\left\|B_i^{-1} C_i\right\| < 1$ and $\|\alpha_i\| < 1$ then $\|R_i\| \equiv \left\|B_i^{-1} C_i \alpha_i\right\| < 1$ and we can apply Lemma 1:

$$\begin{aligned} \|\alpha_{i-1}\| &\le \frac{1}{1 - \|R_i\|} \left\|B_i^{-1} A_i\right\| \\ &\le \frac{1}{1 - \left\|B_i^{-1} C_i\right\| \|\alpha_i\|} \left\|B_i^{-1} A_i\right\| \\ &\le \frac{1}{1 - \left\|B_i^{-1} C_i\right\|} \left\|B_i^{-1} A_i\right\| . \end{aligned} \tag{1.19}$$

According to (1.13) and (1.14), the right-hand side of (1.19) is less than one, therefore the lemma is proven. □