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Tensor ranks for the inversion of tensor-product binomials

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ABSTRACT

The main result reads: if a nonsingular matrix A of order $n = pq$ is a tensor-product binomial with two factors then the tensor rank of A^{-1} is bounded from above by $\min\{p, q\}$. The estimate is sharp, and in the worst case it amounts to $\sqrt{n}$.

1. Introduction

Tensor decompositions are becoming a subject accumulating research interests in several important fields. Among them one should mention complexity theory for polynomial and matrix computations [1,2], data analysis supported by the Kruskal uniqueness property and Tucker reduction [3,2] and numerous multidimensional applications (see [4]).

Despite an ever-increasing interest and several significant results (for some bibliography see [3,5,2,6,4]), many natural questions about seemingly “simple” cases are still not answered. In this paper we present some nontrivial tensor rank estimates for the inverse matrices.

Consider a matrix A of order $n = pq$ and its tensor-product decomposition

$$A = \sum_{s=1}^{r} U_s \otimes V_s \tag{1}$$

with minimal possible number of terms. In this case we call r a *tensor rank* of A, and write

$$r = \mathrm{tRank}(A).$$

Matrices U_s and V_s are of order p and q, respectively, and $U \otimes V$ means the Kronecker (tensor) product of matrices U and V.

By definition, if $U = [u_{ij}]$ is of order p and $V = [v_{kl}]$ is of order q, then $U \otimes V$ is a matrix of order $n = pq$ with the following block structure:

$$U \otimes V = \begin{bmatrix} u_{11}V & \dots & u_{1p}V \\ \dots & \dots & \dots \\ u_{p1}V & \dots & u_{pp}V \end{bmatrix}.$$

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Of course, the tensor rank of A is calculated for some fixed p and q whose product is the order of A, so maybe a better way of notation would be

$$\mathrm{tRank}(A) = \mathrm{tRank}_{p,q}(A),$$

but here we still skip p and q for the sake of brevity.

It is obvious that a pair of indices (i, k) with $1 \le i \le p$, $1 \le k \le q$ naturally points to a row in A (and in $U \otimes V$), and similarly, another pair (j, l) indicates a column in A. It follows that

$$(A)_{(i,k),(j,l)} = \sum_{s=1}^{r} (U_s)_{ij} (V_s)_{kl}. \tag{2}$$

This suggests reshaping A into a matrix

$$B = \mathrm{reshape}(A)$$

in the following way:

$$(B)_{(i,j),(l,l)} = (A)_{(i,k),(j,l)}, \tag{3}$$

the sizes of A and B being $(pq) \times (pq)$ and $p^2 \times q^2$, respectively. Then, Eqs. (2) and (3) show that B is the sum of r column-by-row matrices. Consequently (see [7]),

$$\mathrm{tRank}(A) = \mathrm{rank}(B). \tag{4}$$

We are interested in sharp estimates for the tensor rank of A^{-1}, provided that A is nonsingular. A trivial bound emanating from (4) reads

$$\mathrm{tRank}(A^{-1}) \le n,$$

and does not reveal any dependence on r, or on p and q. The results presented below assume that the entries of all matrices are complex numbers or belong to a subfield of complex numbers; whether they are valid for finite fields is an open question.

2. Main result

Let $n = pq$; the tensor decompositions are considered for fixed p and q. Assume that $A = U \otimes V$ is nonsingular. Then both U and V are nonsingular and, as is easy to check,

$$(U \otimes V)^{-1} = U^{-1} \otimes V^{-1}.$$

Therefore, if $\mathrm{tRank}(A) = 1$, then $\mathrm{tRank}(A^{-1}) = 1$.

All the cases with $\mathrm{tRank}(A) \ge 2$ are significantly more intricate.

Theorem 2.1. *Let a matrix A of order $n = pq$ be nonsingular, and assume that*

$$\mathrm{tRank}(A) = 2.$$

Then

$$\mathrm{tRank}(A^{-1}) \le \min\{p, q\}.$$

Proof. Under the premises of the theorem we can write

$$A = U_1 \otimes V_1 + U_2 \otimes V_2.$$

Let us assume first that all the matrices U_1, V_1, U_2, V_2 are nonsingular. Moreover, assume that all the eigenvalues of $U_1 U_2^{-1}$ and $V_2 V_1^{-1}$ are simple. Then both matrices can be diagonalized by similarity transformations:

$$U_1 U_2^{-1} = X \Lambda X^{-1}, \quad \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\},$$
$$V_2 V_1^{-1} = Y \mathrm{M} Y^{-1}, \quad \mathrm{M} = \mathrm{diag}\{\mu_1, \ldots, \mu_q\}.$$

Hence,

$$A\,(U_2^{-1} \otimes V_1^{-1}) = (X \otimes Y)\, Z\, (X^{-1} \otimes Y^{-1}),$$

where

$$Z = \Lambda \otimes I + I \otimes \mathrm{M}.$$