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Fixed point solutions of generalized equilibrium problems for nonexpansive mappings

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ABSTRACT

In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space. Then, strong convergence of the scheme to a common element of the two sets is proved. As an application, problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem is solved. Moreover, solution is given to the problem of finding a common element of fixed points set of nonexpansive mappings and the set of solutions of a variational inequality problem.

1. Introduction

Let K be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in K. \tag{1.1}$$

The variational inequality problem is to find an $x^* \in K$ such that

$$\langle y - x^*, Ax^*\rangle \geq 0, \quad \forall y \in K. \tag{1.2}$$

(See e.g., [1–3].) We shall denote the set of solutions of the variational inequality problem (1.2) by $VI(K, A)$.

A mapping $A : K \to H$ is called *inverse-strongly monotone* (see e.g., [2,4]) if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, $\forall x, y \in K$. For such a case, A is called α-inverse-strongly monotone.

A mapping $T : K \to K$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K.$$

A mapping $T : K \to K$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in K$. If $k = 0$, then the mapping T is nonexpansive. Observe that if T is a k-strictly pseudocontractive and we put $A := I - T$, where I is the identity operator defined on K, then we have that

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2$$

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for all $x, y \in K$ and since H is a real Hilbert space, we have that

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 + \|Ax-Ay\|^2 - 2\langle x-y, Ax-Ay\rangle.$$

So,

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^2.$$

Thus, if T is a k-strictly pseudocontractive mapping, then $A = I - T$ is an α-inverse-strongly monotone operator with $\alpha = \frac{1-k}{2}$.

A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in K : Tx = x\}$.

Let F be a bifunction of $K \times K$ into $\mathbb{R}$, the set of reals and $A : K \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in K$ such that

$$F(x,y) + \langle Ax, y-x\rangle \geq 0, \tag{1.3}$$

for all $y \in K$. The set of solutions of this generalized equilibrium problem is denoted by EP. Thus

$$EP := \{x^* \in K : F(x^*, y) + \langle Ax^*, y-x^*\rangle \geq 0, \forall y \in K\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(F)$ and in the case of $F \equiv 0$, EP is denoted by $VI(K, A)$. The problem (1.3) includes as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, etc. (see, for example, [5,6]).

Recently, Takahashi and Takahashi [7] introduced an iterative scheme for approximating the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space. In particular, they proved the following theorem.

Theorem 1.1. *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $K \times K$ satisfying* (A1)–(A4)*, A be an α-inverse-strongly monotone mapping of K into H and let T be a nonexpansive mapping of K into itself. Suppose $F(T)\bigcap EP \neq \emptyset$ and $u \in K$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are generated by $x_1 \in K$,*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \quad \forall y \in K \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T[\alpha_n u + (1-\alpha_n)z_n], \quad n \geq 1; \end{cases} \tag{1.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$. If $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty,$$
$$\lim_{n\to\infty}|r_{n+1} - r_n| = 0, \qquad 0 < c \leq \beta_n \leq d < 1$$

then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_0 = P_{F(T)\bigcap EP}u$.

Motivated by (1.4), we introduce *an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space*. We show that the iterative scheme converges strongly to a common element of the two sets. As an application, problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem is solved. Moreover, solution is given to the problem of finding a common element of fixed points set of a nonexpansive mappings and the set of solutions of a variational inequality problem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle ., .\rangle$ and norm $\|.\|$ and let K be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written $x_n \to x$ as $n \to \infty$.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - y\|, \quad \forall y \in K.$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2, \quad \text{for all } x, y \in H. \tag{2.1}$$

Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y\rangle \geq 0, \tag{2.2}$$