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Vector continuous-time programming without differentiability

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ABSTRACT

In this work continuous-time programming problems of vector optimization are considered. Firstly, a nonconvex generalized Gordan's transposition theorem is obtained. Then, the relationship with the associated weighting scalar problem is studied and saddle point optimality results are established. A scalar dual problem is introduced and duality theorems are given. No differentiability assumption is imposed.

1. Introduction

The following vector continuous-time programming problem is addressed:

$$\begin{aligned} &\text{minimize} && \phi(x) = \int_0^T f(x(t), t)\mathrm{d}t \\ &\text{subject to} && g(x(t), t) \leq 0 \quad \text{a.e. in } [0, T], \\ &&& x \in X. \end{aligned} \tag{VCP}$$

Here X is a nonempty subset of the Banach space $L_\infty^n[0, T]$, $\phi : X \to \mathbb{R}^p$, $f(x(t), t) = \xi(x)(t)$, $g(x(t), t) = \gamma(x)(t)$, where $\xi : X \to \Lambda_1^p[0, T]$, and $\gamma : X \to \Lambda_1^m[0, T]$. $L_\infty^n[0, T]$ denotes the space of all n-dimensional vector valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0, T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined as

$$\|x\|_\infty = \max_{1\leq j\leq n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \leq t \leq T\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the j-th component of $x(t) \in \mathbb{R}^n$ and $\Lambda_1^m[0, T]$ denotes the space of all m-dimensional vector-valued functions which are essentially bounded and Lebesgue measurable, defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined as

$$\|y\|_1 = \max_{1\leq j\leq m} \int_0^T |y_j(t)|\mathrm{d}t.$$

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Continuous-time problems were introduced in Bellman [1] in connection with production-inventory "bottleneck processes", where he considered a continuous-time linear programming problem. In 1965 Tyndall [2] generalized Bellman's initial formulations by giving mathematically rigorous treatment to the problem, in the linear case. Nonlinear problems were first investigated, in 1968 by Hanson [3] and Hanson and Mond [4] and in 1974 by Farr and Hanson [5]. Since then an extensive bibliography has been produced. For more information and literature on continuous-time problems, I cite [6,7] and the references cited therein, for smooth problems. For nonsmooth problems, I cite [8–10]. Multiobjective programs were studied, for example, in [11–15].

The applicability and importance of the concepts of convexity, generalized convexity, invexity and generalized invexity on getting optimality conditions and duality results in mathematical programming and optimization problems are well known. In the case of continuous-time problems, see, for instance, [6,10,12,13,16].

This paper will make use of the preinvexity notion. Preinvex functions were introduced in Hanson and Mond [17] as a generalization of non-differentiable invex functions. Preinvex functions can also be seen as functions which possess convex like properties over invex sets. Afterwards, Weir and Jeyakumar [18] and Weir and Mond [19] worked with preinvex functions in scalar-valued and vector-valued programs. L. Batista dos Santos et al. [20] defined preinvex functions in Banach spaces and obtained optimality conditions for vector abstract optimization problems.

In [21] a generalization of the Gordan's Theorem of the alternative (see [22]) in the continuous-time context is developed. Such theorem is valid for convex functions. In this work (in Section 3) a version of the Generalized Gordan's Theorem for preinvex functions is presented. The theorem is then used to prove other results in the subsequent sections.

In Section 4 the Geoffrion [23] scheme is applied for (VCP); i.e., the correlation between (VCP) and the associated weighting scalar problem is established.

Saddle point optimality for convex continuous-time problems is studied in [7,15,24,25]. Here (in Section 5) preinvex problems are considered.

Osuna-Gómez et al. [26] introduced a new dual problem for vector optimization problems with the special feature of being a scalar program. In this paper (see Section 6) a similar one for continuous-time problems is defined and duality theorems are proved.

Next some preliminaries are provided, that is, some notations and definitions common for the whole article.

2. Preliminaries

In this paper, all vectors are column vectors and a prime is used to denote transposition.

Let us denote by C the positive closed orthant of $\mathbb{R}^k$, so that given $u, v \in \mathbb{R}^k$, $u \geq v$ means $u - v \in C$ and $u > v$ means $u - v \in \text{int}(C)$.

Let F be the set of all feasible solutions of problem (VCP) (which is supposed to be nonempty); i.e.,

$$F = \{x \in X : g(x(t), t) \leq 0 \text{ a.e. in } [0, T]\}.$$

Definition 2.1. A feasible solution $\bar{x}$ is said to be an *efficient solution* of (VCP) if there does not exist another feasible solution x such that $\phi(x) \leq \phi(\bar{x})$ and $\phi(x) \neq \phi(\bar{x})$.

Let

$$\phi_j(x) = \int_0^T f_j(x(t), t)\mathrm{d}t, \quad x \in X, \ j \in \{1, \dots, p\},$$

where $f_j(x(t), t)$ denotes the j-th component of $f(x(t), t) \in \mathbb{R}^p$.

Definition 2.2. A feasible solution $\bar{x}$ is said to be a *properly efficient solution* of (VCP) if it is efficient and if there exists a scalar $M > 0$ such that, for each i, we have

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{\phi_j(x) - \phi_j(\bar{x})} \leq M$$

for some j such that $\phi_j(x) > \phi_j(\bar{x})$, when $x \in F$ and $\phi_i(x) < \phi_i(\bar{x})$.

Definition 2.3. Let S be a subset of a Banach space E. We say that S is *invex* with respect to $\eta : S \times S \to E$ if for all $x_1, x_2 \in S$ and for each $\lambda \in (0, 1)$,

$$x_2 + \lambda\eta(x_1, x_2) \in S.$$

Definition 2.4. If S is invex with respect to $\eta : S \times S \to \mathbb{R}^n$, a given function $\theta : S \to \mathbb{R}^k$ is called *preinvex* with respect to η if for all $x_1, x_2 \in S$ and for each $\lambda \in (0, 1)$,

$$\theta(x_2 + \lambda\eta(x_1, x_2)) \leq \lambda\theta(x_1) + (1 - \lambda)\theta(x_2).$$