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Letter to the editor

# Numerical differentiation for high orders by an integration method[☆]

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ABSTRACT

This paper mainly studies the numerical differentiation by integration method proposed first by Lanczos. New schemes of the Lanczos derivatives are put forward for reconstructing numerical derivatives for high orders from noise data. The convergence rate of these proposed methods is $O\left(\delta^{\frac{4}{n+4}}\right)$ as the noise level $\delta \to 0$. Numerical examples show that the proposed methods are stable and efficient.



## 1. Introduction

Numerical differentiation provides a way of numerically determining the derivatives of an unknown function from its approximate values; and it is an important method in scientific research and engineering disciplines. For example, solutions related to image processing [1], magnetic resonance electrical impedance tomography [2,3] and identification [4] can be improved if the derivatives are obtained by a high accuracy approximation method. The problem of numerical differentiation is well known to be ill-posed, which means that the small errors in measurement data of the function can induce large errors in its computed derivatives. Therefore, various numerical methods have been suggested for obtaining the numerical derivatives [1,5–11]. These works mainly fall into four types: the mollification methods [5], the finite difference methods [6], the regularization methods [1,7,8] and the differentiation by integration methods [9–11]. The differentiation by integration methods, i.e. using the Lanczos generalized derivatives, are simple and effective methods firstly proposed by Lanczos [9].

The Lanczos generalized derivative [9] $D_h$, defined by

$$D_h f(x) = \frac{3}{2h^3}\int_{-h}^{h} tf(x+t)\mathrm{d}t = \frac{3}{2h}\int_{-1}^{1} tf(x+ht)\mathrm{d}t, \tag{1.1}$$

approximates $f'(x)$ in the sense $f'(x) = D_h f(x) + O(h^2)$. Recently, Rangarajana et al. [11] generalized it to the case for high order derivatives with

$$D_h^{(n)} f(x) = \frac{1}{h^n}\int_{-1}^{1} \rho_n(t) f(x+ht)\mathrm{d}t, \quad n = 1, 2, \ldots, \tag{1.2}$$

which is an approximation of the $n$th-order derivative $f^{(n)}(x)$ and obtained by choosing $\rho_n(t)$ such that

$$D_h^{(n)} f(x) = f^{(n)}(x) + O(h^2). \tag{1.3}$$

In fact, it is shown in [11] that $\rho_n(t)$ is proportional to the Legendre polynomial $P_n(t)$ by Taylor expansion, namely $\rho_n(t) = \gamma_n P_n(t)$, where $P_n(t)$ is the $n$th-order Legendre polynomial and $\gamma_n = \frac{1\times 3\times 5\times\cdots\times(2n+1)}{2}$.

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In this paper, we continue to work on the differentiation by integration methods to propose new schemes for obtaining the Lanczos derivatives for high orders. This paper is organized as follows: New schemes for obtaining the Lanczos derivatives for high orders are proposed with the corresponding convergence rate in Section 2. Numerical examples are given in Section 3 for verifying the efficiency and stability of the proposed schemes.

## 2. New schemes for obtaining high order derivatives

Introducing the following operator denoted by $D_h^{(n)}f$:

$$D_h^{(n)}f(x) = \frac{1}{h^n}\int_{-1}^{1} P_n(t)[\alpha_n f(x+ht) + \beta_n f(x+\lambda_n ht)]\mathrm{d}t, \tag{2.1}$$

where $P_n(t)$ is the $n$th-order Legendre polynomial, we choose $\alpha_n$, $\beta_n$, $\lambda_n$ such that

$$D_h^{(n)}f(x) = f^{(n)}(x) + O(h^4). \tag{2.2}$$

Since $P_n(-t) = (-1)^n P_n(t)$, we conclude that $D_h^{(n)}f(x) = D_{-h}^{(n)}f(x)$. So, we always take $h$ to be positive in the following.

To get the rate of convergence of (2.1), we assume that $f(x)$ is bounded and has a continuous $(n+4)$th derivative on some interval $I$ containing the points $x$, $x \pm h$, $x \pm \lambda_n h$. Assume further that $f^\delta(x)$ is some bounded integrable approximation of $f(x)$ satisfying

$$\|f^\delta(x) - f(x)\|_\infty = \sup_{x\in I}|f^\delta(x) - f(x)| \le \delta, \tag{2.3}$$

where $\delta$ is the noise level. To determine the coefficients $\alpha_n$, $\beta_n$, $\lambda_n$, we write the Taylor expansion for a given $n$ as follows:

$$\begin{aligned} f(x+ht) &= f(x) + htf'(x) + \cdots + \frac{h^n t^n}{n!}f^{(n)}(x) + \frac{h^{n+1}t^{n+1}}{(n+1)!}f^{(n+1)}(x) \\ &\quad + \frac{h^{n+2}t^{n+2}}{(n+2)!}f^{(n+2)}(x) + \frac{h^{n+3}t^{n+3}}{(n+3)!}f^{(n+3)}(x) + \frac{h^{n+4}t^{n+4}}{(n+4)!}f^{(n+4)}(\xi). \end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.1) and noting the orthogonal property of Legendre polynomials, we know that

$$\frac{\alpha_n}{k!}\int_{-1}^{1} t^k P_n(t) + \frac{\beta_n\lambda_n^k}{k!}\int_{-1}^{1} t^k P_n(t)\mathrm{d}t = 0, \quad \forall k < n. \tag{2.5}$$

Furthermore, noting that

$$\frac{\alpha_n}{(k)!}\int_{-1}^{1} t^k P_n(t) + \frac{\beta_n\lambda_n^k}{(k)!}\int_{-1}^{1} t^k P_n(t)\mathrm{d}t = 0, \quad k = n+1, n+3, \tag{2.6}$$

we require that $\alpha_n$, $\beta_n$, $\lambda_n$ satisfy

$$\frac{\alpha_n}{n!}\int_{-1}^{1} t^n P_n(t)\mathrm{d}t + \frac{\beta_n\lambda_n^n}{n!}\int_{-1}^{1} t^n P_n(t)\mathrm{d}t = 1, \tag{2.7}$$

$$\frac{\alpha_n}{(n+2)!}\int_{-1}^{1} t^{n+2} P_n(t)\mathrm{d}t + \frac{\beta_n\lambda_n^{n+2}}{(n+2)!}\int_{-1}^{1} t^{n+2} P_n(t)\mathrm{d}t = 0. \tag{2.8}$$

Let $p_n = \int_{-1}^{1} t^n P_n(t)\mathrm{d}t$. From (2.7) and (2.8), we obtain

$$\begin{cases} \alpha_n + \beta_n\lambda_n^n = \dfrac{n!}{p_n}; \\ \alpha_n + \beta_n\lambda_n^{n+2} = 0. \end{cases} \tag{2.9}$$

By solving the system (2.9), we get

$$\alpha_n = -\frac{n!\lambda_n^2}{p_n(1-\lambda_n^2)}, \qquad \beta_n = \frac{n!}{p_n(\lambda_n^n - \lambda_n^{n+2})}. \tag{2.10}$$

From (2.10), $\lambda_n$ is a free parameter. That is to say, there are infinitely varied schemes of (2.1) for computing the $n$th-order derivative. Does there exist an optimal $\lambda_n$? We will answer this problem in the sequel from the viewpoint of computation. Since $\int_{-1}^{1} t^{n+3}P_n(t)\mathrm{d}t = 0$, we have the following theorem from (2.4)–(2.8).