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A simple finite volume method for the shallow water equations

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ABSTRACT

We present a new finite volume method for the numerical solution of shallow water equations for either flat or non-flat topography. The method is simple, accurate and avoids the solution of Riemann problems during the time integration process. The proposed approach consists of a predictor stage and a corrector stage. The predictor stage uses the method of characteristics to reconstruct the numerical fluxes, whereas the corrector stage recovers the conservation equations. The proposed finite volume method is well balanced, conservative, non-oscillatory and suitable for shallow water equations for which Riemann problems are difficult to solve. The proposed finite volume method is verified against several benchmark tests and shows good agreement with analytical solutions.

1. Introduction

During the last decades there has been an enormous amount of activity related to the construction of approximate solutions for the shallow water equation written in conservative form as

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}, \tag{1.1}$$

where $Z(x)$ is the function characterizing the bottom topography, $h(t, x)$ is the height of the water above the bottom, g is the acceleration due to gravity and $u(t, x)$ is the flow velocity. Eq. (1.1) has been widely used to model water flows, flood waves, dam-break problems, and has been studied in a number of books and papers; compare [1–5] among others. Computing their numerical solutions is not trivial due to nonlinearity, the presence of the convective term and the coupling of the equations through the source term. In many applications of (1.1), the convective terms are distinctly more important than the source terms; particularly when certain non-dimensional parameters reach high values (e.g. the Froude number), these convective terms are a source of computational difficulties and oscillations. It is well known that the solutions of Eq. (1.1) present steep fronts and even shock discontinuities, which need to be resolved accurately in applications and often cause severe numerical difficulties [6,2].

Many numerical methods are available in the literature to solve the shallow water equations. One of the most popular techniques is the well-known Roe scheme [7] originally designed for hyperbolic systems without accounting for source terms. In [8], the authors modified the Roe scheme [7] to solve the shallow water equations with source terms in which the idea of balancing the gradient flux with the source term is formulated. This method has been improved in [9] for general channel flows. However, for practical applications, this method may become computationally demanding due to its treatment of the source terms. In the context of well-balanced methods, we mention the work [10] developed to analyze

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the source term due to cross-section irregularities, and the work in [11] which analyzes the effects of source term in flux difference splitting technique. The authors in [12] have developed exact solutions for the Riemann problem at the interface with a sudden variation in the topography. The main idea in their approach was to define the bottom level such that a sudden variation in the topography occurs at the interface of two cells. A different approach was adopted in quasi-steady wave propagation method in [6]. In this method, an additional Riemann problem in the center of each cell is introduced for balancing the source terms and the flux gradients. An approach based on a local hydrostatic reconstruction has been proposed in [13] for open channel flows with topography. The extension of ENO and WENO schemes to shallow water equations has been studied in [14]. Unfortunately, most ENO and WENO schemes that solves real flows correctly are still very computationally expensive. In the framework of Runge–Kutta discontinuous Galerkin methods, authors in [15] extended the method to a class of hyperbolic system of balance laws with separable source terms. The central idea in this approach is a proper decomposition of the source term allowing well balancing and preserving the genuinely high resolution of the method. However, most of these methods present results with an order of accuracy smaller than the expected in the solutions for unstructured grids, see for example [16]. Besides this fact, it is well known that TVD schemes have their order of accuracy reduced to first order in the presence of shocks due to the effects of limiters. On the other hand, numerical methods based on kinetic reconstructions have been studied in [17], but the complexity of these methods is relevant.

In current work we propose a new family of numerical schemes that incorporate the techniques from method of characteristics into the reconstruction of numerical fluxes. Our main goal is to present a class of numerical methods that are simple, easy to implement, and accurately solves the shallow water equations without relying on a Riemann solver. This goal is reached by integrating twice the shallow water system (1.1) in time and space. In the first integration, Eq. (1.1) is integrated over an Eulerian time–space control volume. We term this step by corrector stage applied to the conservation equations. In the second integration, the shallow water equations are rewritten in a non-conservative form and integrated along the characteristics defined by the water velocity. This step is called predictor stage and used to calculate the numerical fluxes required in the corrector stage. Our method can be treated as a conservative modified method of characteristics for shallow water equations or as a Riemann solver-free finite volume method for shallow water equations. To approximate the characteristic curves an iterative process is used and numerical fluxes are computed by using interpolation procedures. The discretization of flux gradients and source terms are well balanced and the method satisfies the exact C-property. The proposed scheme has the ability to handle calculations of slowly varying flows as well as rapidly varying flows over continuous and discontinuities bottom beds. We should mention that another advantage in using the method of characteristic is that no boundary conditions are needed for the numerical fluxes at the predictor stage. These features are demonstrated using several benchmark problems for shallow water flows. Results presented in this paper show high resolution of the proposed finite volume characteristics method and confirm its capability to provide accurate and efficient simulations for shallow water flows including complex topography.

In this paper, first the finite volume characteristics method is formulated in Section 2. Thereafter, an analysis of stability and convergence is presented in Section 3. Section 4 is devoted to the application of our method to the shallow water equations. After numerical results and examples are presented in Section 5, accuracy and efficiency of the finite volume characteristics scheme are discussed. Concluding remarks end the paper in Section 6.

2. The finite volume characteristics method

To formulate the finite volume characteristics (FVC) scheme we first consider a scalar homogeneous equation of a nonlinear conservation law given by

$$\partial_t u + \partial_x f(u) = 0. \tag{2.1}$$

We discretize the space domain in cells $[x_{i-1/2}, x_{i+1/2}]$ with same length Δx for sake of simplicity. We also divide the time interval into subintervals $[t_n, t_{n+1}]$ with uniform size Δt. Here, $t_n = n\Delta t$, $x_{i-1/2} = i\Delta x$ and $x_i = (i + 1/2)\Delta x$ is the center of the control volume. Integrating Eq. (2.1) with respect to time and space over the time–space control domain $[t_n, t_{n+1}] \times [x_{i-1/2}, x_{i+1/2}]$ shown in Fig. 2.1, we obtain the following discrete equation

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(f(U_{i+1/2}^n) - f(U_{i-1/2}^n)\right), \tag{2.2}$$

where U_i^n is the space average of the solution u in the control volume $[x_{i-1/2}, x_{i+1/2}]$ at time t_n i.e.,

$$U_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(t_n, x)\, dx,$$

and $f(U_{i\pm1/2}^n)$ are the numerical fluxes at $x = x_{i\pm1/2}$ and time t_n. The spatial discretization of Eq. (2.2) is complete when a numerical construction of the fluxes $f(U_{i\pm1/2}^n)$ is chosen. In general, this construction requires a solution of Riemann problems at the interfaces $x_{i\pm1/2}$. From a computational viewpoint, this procedure is very demanding and may restrict the application of the method for which Riemann solutions are not available.

In the present work, we reconstruct the intermediate states $U_{i\pm1/2}^n$ using the method of characteristics. The fundamental idea of this method is to impose a regular grid at the new time level and to backtrack the flow trajectories to the previous