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Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems[☆]

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ABSTRACT

We propose and study a posteriori error estimates for convection–diffusion–reaction problems with inhomogeneous and anisotropic diffusion approximated by weighted interior-penalty discontinuous Galerkin methods. Our twofold objective is to derive estimates without undetermined constants and to analyze carefully the robustness of the estimates in singularly perturbed regimes due to dominant convection or reaction. We first derive locally computable estimates for the error measured in the energy (semi)norm. These estimates are evaluated using $\mathbf{H}(\mathrm{div}, \Omega)$-conforming diffusive and convective flux reconstructions, thereby extending the previous work on pure diffusion problems. The resulting estimates are semi-robust in the sense that local lower error bounds can be derived using suitable cutoff functions of the local Péclet and Damköhler numbers. Fully robust estimates are obtained for the error measured in an augmented norm consisting of the energy (semi)norm, a dual norm of the skew-symmetric part of the differential operator, and a suitable contribution of the interelement jumps of the discrete solution. Numerical experiments are presented to illustrate the theoretical results.

1. Introduction

We consider the convection–diffusion–reaction problem

$$-\nabla \cdot (\mathbf{K}\nabla u) + \boldsymbol{\beta} \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polyhedral domain, $\mathbf{K}$ the diffusion tensor, $\boldsymbol{\beta}$ the velocity field, μ the reaction coefficient, and f the source term. We only consider homogeneous Dirichlet boundary conditions for the sake of simplicity; extensions to inhomogeneous Dirichlet and Neumann boundary conditions are possible. Our intention is to study a posteriori error estimates for the approximation of (1a)–(1b) by weighted interior-penalty discontinuous Galerkin (DG) methods with the twofold objective of deriving estimates without undetermined constants and analyzing carefully the robustness of

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the estimates in singularly perturbed regimes due to dominant convection or reaction. We have chosen to address the convection–diffusion–reaction problem in a general setting for the parameters $\mathbf{K}$, $\boldsymbol{\beta}$, and μ (mild assumptions on these parameters are stated below) so that our results can be readily used in practical simulations. The reader interested in simplified situations can for instance take $\mathbf{K}$ equal to ϵ times the identity matrix ($\epsilon \ll 1$), $\boldsymbol{\beta}$ a divergence-free velocity field of order unity, and μ of order unity.

For the pure diffusion problem ((1a)–(1b) with $\boldsymbol{\beta} = \mu = 0$), residual-based a posteriori energy (semi)norm error estimates for DG methods can be traced back to [1,2]; see also [3] for a unified analysis. Although the estimates derived therein are both reliable (that is, they yield an upper bound on the difference between the exact and approximate solution) and locally efficient (that is, they give local lower bounds for the error as well), they feature various undetermined constants. This shortcoming has been remedied recently in [4] upon introducing estimators based on equilibrated fluxes (for the first-order symmetric interior-penalty DG scheme in the case $d = 2$). Such estimates can be reformulated upon introducing a reconstructed $\mathbf{H}(\mathrm{div}, \Omega)$-conforming diffusive flux, say $\mathbf{t}_h$, associated with the approximate DG diffusive flux $-\mathbf{K}\nabla_h u_h$ [5–8]; see also the research report [9]. We also mention [10] where numerical experiments for similar estimators are presented. Error estimates for continuous finite element methods using reconstructed $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes can be traced back to the seminal work of Prager and Synge [11], while more recent developments include [12–14].

A posteriori error estimates based on flux reconstruction for DG approximations to *convection–diffusion–reaction* problems appear to be a novel topic. Our first intermediate, yet practically important, result delivers a locally computable, global upper bound for the error measured in the energy (semi)norm $|||\cdot|||$ defined by Eq. (5). Letting u be the exact solution of (1a)–(1b) and letting u_h be its DG approximation, Theorem 3.1 states that

$$|||u - u_h||| \le \eta,$$

where η collects various locally computable contributions with only known constants, the leading terms for low enough local Péclet numbers having constant equal to one. These contributions are evaluated using a $H_0^1(\Omega)$-conforming reconstruction of the potential u_h and $\mathbf{H}(\mathrm{div}, \Omega)$-conforming reconstructions of its diffusive flux $-\mathbf{K}\nabla_h u_h$ and convective flux $\boldsymbol{\beta} u_h$, thereby extending previous work on pure diffusion problems. Theorem 3.2 then states that the elementwise contributions in η can be bounded by the local error in the energy (semi)norm augmented by the natural DG jump seminorm $|||\cdot|||_{*,\mathcal{F}_h}$ defined by (45) times suitable cutoff functions of the local Péclet and Damköhler numbers. More precisely, this yields

$$\eta \le C\chi(|||u - u_h||| + |||u - u_h|||_{*,\mathcal{F}_h}),$$

where the constant C is independent of any mesh size and mildly depends on the data $\mathbf{K}$, $\boldsymbol{\beta}$, and μ as specified below, whereas χ collects the above-mentioned cutoff functions. This result is in its form similar to that derived by Verfürth for stabilized conforming finite elements in [15] and to the results in [16–18] for DG, mixed finite element, and finite volume methods, respectively. The difference with [15] is that the present η features no undetermined constant. Moreover, η represents a lower bound for the DG residual-based a posteriori estimate derived in [16].

To achieve full robustness in singularly perturbed regimes resulting from dominant advection or reaction, we follow the approach proposed again by Verfürth for stabilized conforming finite elements in [19] and which consists in measuring the error in an augmented norm including a suitable dual norm of the skew-symmetric part of the differential operator. Another approach to robust a posteriori error estimation has been proposed by Sangalli [20–22]; it consists in evaluating the convective derivative using a fractional order norm. For DG methods, the augmented norm $|||\cdot|||_{\oplus}$ defined by (46) differs from that considered in the conforming case and features an additional contribution which depends on the interelement jumps of the discrete solution. By proceeding this way, see Theorem 3.3, an upper bound is derived in the form

$$|||u - u_h|||_{\oplus} \le \widetilde{\eta},$$

where $\widetilde{\eta}$ again collects various locally computable contributions (with only known constants as for η) which are evaluated using the above-mentioned reconstructions. Theorem 3.4 then states that $\widetilde{\eta}$ can be globally bounded by the error measured in the augmented norm supplemented by a suitable jump seminorm $|||\cdot|||_{\#,\mathcal{F}_h}$ defined by Eq. (51), that is,

$$\widetilde{\eta} \le \tilde{C}(|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h}),$$

where the constant $\tilde{C}$ has dependencies similar to those of the constant C above. By adding this jump seminorm to the error measure as well, we arrive at the final result of this paper, see Theorem 3.5, yielding a fully robust equivalence result between the error and the a posteriori estimate, namely

$$|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h} \le \widetilde{\eta} + |||u_h|||_{\#,\mathcal{F}_h} \le \tilde{C}(|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h}).$$

This result is in its form similar to the one derived recently in [23] for DG methods using residual-based techniques instead of flux reconstruction. However, there are two important differences between the present results and those in [23]. First, the latter contain undetermined constants; furthermore, the present jump seminorm features an additional cutoff function to lower its contribution in the singularly perturbed regimes.

This paper is organized as follows. We introduce the setting in Section 2, including the main notation and assumptions, the formulation of the continuous problem and its DG approximation, the reconstructed $\mathbf{H}(\mathrm{div}, \Omega)$-conforming diffusive and convective fluxes for the DG solution, and the cutoff functions needed to formulate our results. We then present our