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Adaptive stochastic weak approximation of degenerate parabolic equations of Kolmogorov type

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ABSTRACT

Degenerate parabolic equations of Kolmogorov type occur in many areas of analysis and applied mathematics. In their simplest form these equations were introduced by Kolmogorov in 1934 to describe the probability density of the positions and velocities of particles but the equations are also used as prototypes for evolution equations arising in the kinetic theory of gases. More recently equations of Kolmogorov type have also turned out to be relevant in option pricing in the setting of certain models for stochastic volatility and in the pricing of Asian options. The purpose of this paper is to numerically solve the Cauchy problem, for a general class of second order degenerate parabolic differential operators of Kolmogorov type with variable coefficients, using a posteriori error estimates and an algorithm for adaptive weak approximation of stochastic differential equations. Furthermore, we show how to apply these results in the context of mathematical finance and option pricing. The approach outlined in this paper circumvents many of the problems confronted by any deterministic approach based on, for example, a finite-difference discretization of the partial differential equation in itself. These problems are caused by the fact that the natural setting for degenerate parabolic differential operators of Kolmogorov type is that of a Lie group much more involved than the standard Euclidean Lie group of translations, the latter being relevant in the case of uniformly elliptic parabolic operators.

1. Introduction

The simplest form of an operator of Kolmogorov type is the following degenerate parabolic operator in $\mathbb{R}^{2n+1}$,

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=n+1}^{2n} x_{j-n} \frac{\partial}{\partial x_j} - \partial_t. \tag{1.1}$$

The operator in (1.1) was introduced by Kolmogorov in 1934 in order to describe the density of a system with $2n$ degrees of freedom. In particular, here $\mathbb{R}^{2n}$ represents the phase space where $(x_1, \ldots, x_n)$ and $(x_{n+1}, \ldots, x_{2n})$ are, respectively, the velocity and position of the system; see [1]. An area of applied mathematics where operators of Kolmogorov type recently have turned out to be relevant is that of mathematical finance and option pricing. Degenerate equations of Kolmogorov type arise naturally in the problem of pricing path-dependent contingent claims referred to as Asian-style derivatives; see [2–4] and the references therein. In particular, after some manipulations the pricing of a geometric average Asian option in the standard Black–Scholes model is equivalent to solving the Cauchy problem for the operator (1.1), in this case $n = 1$, in $\mathbb{R}^2 \times [0, T]$ with Cauchy data, also called terminal data, defined by the pay-off of the contract. Moreover, the Cauchy problem

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for operators of Kolmogorov type, more general than that stated in (1.1) and with variable coefficients, also appears in the pricing of general European derivatives in the framework of the stochastic volatility model suggested by Hobson and Rogers; see [3,5].

The purpose of this paper is to apply and work out, for the backward in time Cauchy problem for a general class of second order degenerate parabolic partial differential operators of Kolmogorov type, the approach concerning a posteriori error estimates and adaptive weak approximations of stochastic differential equation due to Szepessy, Tempone and Zouraris, see [6]. Furthermore, we show how this approach can be applied to problems in mathematical finance and option pricing where degenerate parabolic operators of Kolmogorov type occur. In particular, we consider operators of the form

$$L=\frac{1}{2}\sum_{i,j=1}^{m}[\sigma\sigma^*]_{ij}(x,t)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{m}b_i(x,t)\frac{\partial}{\partial x_i}+\sum_{i,j=1}^{n}c_{ij}x_i\frac{\partial}{\partial x_j}+\frac{\partial}{\partial t} \tag{1.2}$$

where $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, m is a positive integer satisfying $m\le n$, $\sigma(x,t)=\{\sigma_{ij}(x,t)\}:\mathbb{R}^n\times\mathbb{R}_+\to M(n,m,\mathbb{R})$, $M(n,m,\mathbb{R})$ is the set of all $n\times m$ matrices with real valued entries and σ^* is the transpose of the matrix σ. $[\sigma\sigma^*]_{ij}(x,t)$ denotes the (i,j) entry of the matrix $[\sigma\sigma^*](x,t)$. The functions $\{\sigma_{ij}(\cdot,\cdot)\}$ and $\{b_i(\cdot,\cdot)\}$ are continuous with bounded derivatives and the matrix $C:=\{c_{ij}\}$ is a matrix of constant real numbers. Note that we are particularly interested in the case $m<n$. Given $T>0$ we consider the problem

$$\begin{cases}Lu(x,t)=0 & \text{whenever } (x,t)\in\mathbb{R}^n\times(0,T),\\ u(x,T)=g(x) & \text{whenever } x\in\mathbb{R}^n,\end{cases} \tag{1.3}$$

where g is a given function. The problem in (1.3) represents the backward in time Cauchy problem for the operator L with terminal data g. Concerning structural assumptions on the operator L we assume that

$$A(x,t)=\{a_{ij}(x,t)\},\quad a_{ij}(x,t):=[\sigma\sigma^*]_{ij}(x,t),\ \text{is symmetric}, \tag{1.4}$$

and that there exists a $\epsilon\in[1,\infty)$ such that

$$\epsilon^{-1}|\xi|^2\le\sum_{i,j=1}^{m}a_{ij}(x,t)\xi_i\xi_j\le\epsilon|\xi|^2\quad\text{whenever } (x,t)\in\mathbb{R}^{n+1},\ \xi\in\mathbb{R}^m. \tag{1.5}$$

Note that in (1.5) we are only assuming ellipticity in m out of n spatial directions. Let $\bar{A}(x,t)=\{\bar{a}_{ij}(x,t)\}$ denote, whenever $(x,t)\in\mathbb{R}^{n+1}$, the unique $m\times m$ matrix which satisfies $\bar{A}(x,t)\bar{A}(x,t)=A(x,t)$. For $(x_0,t_0)\in\mathbb{R}^{n+1}$, fixed but arbitrary, we introduce the differential operators

$$X_0=\sum_{i,j=1}^{n}c_{ij}x_i\frac{\partial}{\partial x_j}+\frac{\partial}{\partial t},\qquad X_i=\frac{1}{\sqrt{2}}\sum_{j=1}^{m}\bar{a}_{ij}(x_0,t_0)\frac{\partial}{\partial x_j},\quad i\in\{1,\dots,m\}, \tag{1.6}$$

as well as the operator

$$\tilde{L}=\tilde{L}_{(x_0,t_0)}:=\sum_{i=1}^{m}X_i^2+X_0=\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(x_0,t_0)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i,j=1}^{n}c_{ij}x_i\frac{\partial}{\partial x_j}+\frac{\partial}{\partial t}. \tag{1.7}$$

To compensate for the lack of ellipticity, see (1.5), we assume that

$$\tilde{L}=\tilde{L}_{(x_0,t_0)}\ \text{is hypoelliptic for every fixed } (x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}_+. \tag{1.8}$$

Let $\mathrm{Lie}(X_0,X_1,\dots,X_m)$ denote the Lie algebra generated by the vector fields $X_0,X_1,\dots,X_m$. It is well known that (1.8) can be stated in terms of the following Hörmander condition:

$$\mathrm{rank\ Lie}(X_0,X_1,\dots,X_m)=n+1\ \text{at every point } (x,t)\in\mathbb{R}^{n+1}. \tag{1.9}$$

Another condition, equivalent to (1.8) and (1.9), is that there exists a basis for $\mathbb{R}^n$ such that the matrix C has the form

$$\begin{pmatrix} * & C_1 & 0 & \cdots & 0\\ * & * & C_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & C_l\\ * & * & * & \cdots & * \end{pmatrix} \tag{1.10}$$

where C_j, for $j\in\{1,\dots,l\}$, is a $m_{j-1}\times m_j$ matrix of rank m_j, $1\le m_l\le\cdots\le m_1\le m_0$ and $m_0+m_1+\cdots+m_l=n$ while $*$ represents arbitrary matrices with constant entries. For a proof of the equivalence between the conditions stated above we refer to [7].