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A linear backward Euler scheme for the saturation equation: Regularity results and consistency ☆

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ABSTRACT

We consider a linearization of a numerical scheme for the saturation equation (or porous medium equation) $\frac{\partial S}{\partial t} - \nabla \cdot f(S)\mathbf{u} - \nabla \cdot k(S)\nabla S = 0$, through first order expansions of the fractional function f and the inverse of the function $K(s) = \int_0^s k(\tau)\mathrm{d}\tau$, after a regularization of the porous medium equation. We establish a regularity result for the Continuous Galerkin Method and a regularity result for the linearized scheme analogous to the corresponding nonlinear scheme. We then show that the linearized scheme is consistent with the nonlinear scheme analyzed in a previous work.

1. Introduction

Consider the saturation equation

$$\begin{cases} \dfrac{\partial S}{\partial t} + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) & \text{on } \Omega \times (0, T_0] \\ (f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = q & \text{on } \partial\Omega \times [0, T_0] \\ S(x, 0) = S^0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

obtained from modeling a two-phase immiscible flow through a porous medium [1–4]. The set Ω is a bounded domain of $\mathbb{R}^n$, $n = 2, 3$. In this work, we have in mind $n = 2$ and Ω a polygonal domain.

The unknown S is the saturation of the invading phase. The diffusion coefficient k is the conductivity of the medium and satisfies the following conditions.

$$k(0) = k(1) = 0, \tag{1.2}$$

$$k(\xi) \geq \begin{cases} c_1\xi^{\mu} & \text{if } 0 \leq \xi \leq \alpha_1 \\ c_2 & \text{if } \alpha_1 < \xi < \alpha_2 \\ c_3(1-\xi)^{\mu} & \text{if } \alpha_2 \leq \xi \leq 1 \end{cases} \tag{1.3}$$

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where $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$, and $0 < \mu \leq 2$. We define K by

$$K(s) = \int_0^s k(\tau)\mathrm{d}\tau. \tag{1.4}$$

The graph of the fractional flow function f is known to have an S-shaped (as a function of the saturation S). So we impose the following conditions on f.

The function f is twice continuously differentiable in the variable s, and

$$f'(0) = f'(1) = 0. \tag{1.5}$$

We notice, by [5,6], that if (1.3) and (1.5) hold, then

$$|f(s_2) - f(s_1)|^2 \leq C|K(s_2) - K(s_1)||s_2 - s_1|. \tag{1.6}$$

We also notice, through (1.6) and (1.4), that

$$|f'(s)| \leq C\sqrt{k(s)}. \tag{1.7}$$

For the purpose of establishing regularity estimates, we assume that $k \in C^1([1, 0])$ and $f \in C^2([0, 1])$.

As in [5,7,6,8], we assume that the Darcy velocity $\mathbf{u}$ is given and has the necessary regularity we need for this analysis.

The main purpose of this paper is to establish regularity estimates for a linearization of the Backward Euler scheme obtained by fully discretizing a regularization of problem (1.1). These regularity results are then used to show that the proposed linearized scheme is consistent, in some sense, with the nonlinear scheme considered in [8]. Problem (1.1) has been studied by many authors under various conditions (see for instance [9–11,6,5,8,12], among others). In the study of the problem, two difficulties (among others) arise: the nonlinearity and the degeneracy of the problem. Because of the degeneracies ($k(0) = k(1) = 0$), Problem (1.1) has often been regularized into a family of nondegenerate problems whose solutions converge to the solution of (1.1) [6,5,8,10,9,12]. Though some studies (see for instance [13]) bypass the regularization step, usually, the numerical approximation of the solution of (1.1) is done in three steps: regularization, Continuous Galerkin Method, and fully discretized Galerkin method. In the last step, some of the works cited above obtain a nonlinear implicit scheme (backward Euler) which needs to be linearized in some way for a computer implementation. Often, a Picard iteration is used (see for instance [14] and [15]).

This paper is a continuation of [7] where a method was proposed that linearizes the scheme. We wish to replace the nonlinear scheme

$$\left(\frac{H_\beta(U_h^{n+1}) - H_\beta(U_h^n)}{\Delta t}, \chi\right) - \left(f(H_\beta(U_h^{n+1}))\mathbf{u}^{n+1}, \nabla\chi\right) + \left(\nabla U_h^{n+1}, \nabla\chi\right) = 0 \tag{1.8}$$

analyzed in [8], (where U_h^n is a discrete approximation of $K(S)$), by the linear scheme (3.1), proposed in [7], through first order expansions of the functions $f \circ H_\beta$ and H_β, where H_β is defined by (2.15). The Taylor expansions considered are:

$$H_\beta(v_2) - H_\beta(v_1) = (v_2 - v_1)H'_\beta(v_1) + O((v_2 - v_1)^2),$$

and

$$(f \circ H_\beta)(v_2) - (f \circ H_\beta)(v_1) = (v_2 - v_1)(f \circ H_\beta)'(v_1) + O((v_2 - v_1)^2).$$

The linearized scheme (3.1) and (3.2), below, is obtained from (1.8), by simply discarding the second and higher order terms in the expansions of H_β and $f \circ H_\beta$.

Using one of the regularity results established here, we show that, in fact, if $v_2 = U_h^{n+1}$ and $v_1 = U_h^n$, then

$$O((v_2 - v_1)^2) = O((\Delta t)^\alpha), \tag{1.9}$$

for some $\alpha > 0$, where the constants intervening in the above estimate are independent of β, the regularization parameter, h, the space discretization parameter, and Δt, the time stepping parameter, though this is true for a balanced choice of β, h, and Δt.

The remainder of the paper is structured as follow.

In Section 2, we state some preliminary results established in previous works.

In Section 3, we state and prove our main results: We prove a new regularity result for the Continuous Galerkin Method and use it to establish the consistency of the linearized scheme. We also prove, for the linearized scheme, a regularity result analogue to some regularity results for the Continuous Galerkin Method and the fully discretized nonlinear scheme established in [8].

Notations. Finally, we set additional notation which will be used throughout the remainder of this paper. We define $(f, g) := (f, g)_\Omega := \int_\Omega fg\mathrm{d}x$ when this has a meaning. The notation $\|f\|_{L^p} := \|f\|_{L^p(\Omega)}$ is used for the standard Lebesgue norm of a measurable function, when this quantity is finite. Similarly, we denote by $\|f\|_{L^p(L^q)} := \|f\|_{L^p(0,T,L^q(\Omega))}$ the mixed Lebesgue norm for f. For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, by $\|\mathbf{v}\|_{L^2}$, we mean $\|\mathbf{v}\|_{(L^2(\Omega))^k}$. We write $V_{ht} := \frac{\partial V_h}{\partial t}$, the partial derivative of V_h with respect to t. Similarly, V_{htt} designates the second order partial derivative of V_h with respect to t. We use C, c, to denote constants which may change from line, but which are independent of the parameters β, h and Δt, unless otherwise explicitly specified. We also denote by $\sigma,\ \sigma_1,\ \sigma_2, \ldots$ etc, constants we can control thanks to the Arithmetic-Geometric Inequality.