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Two new algorithms for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities

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ABSTRACT

This paper studies the optimization model of a linear objective function subject to a system of fuzzy relation inequalities (FRI) with the max-Einstein composition operator. If its feasible domain is non-empty, then we show that its feasible solution set is completely determined by a maximum solution and a finite number of minimal solutions. Also, an efficient algorithm is proposed to solve the model based on the structure of FRI path, the concept of partial solution, and the branch-and-bound approach. The algorithm finds an optimal solution of the model without explicitly generating all the minimal solutions. Some sufficient conditions are given that under them, some of the optimal components of the model are directly determined. Some procedures are presented to reduce the search domain of an optimal solution of the original problem based on the conditions. Then the reduced domain is decomposed (if possible) into several sub-domains with smaller dimensions that finding the components of the optimal solution in each sub-domain is very easy. In order to obtain an optimal solution of the original problem, we propose another more efficient algorithm which combines the first algorithm, these procedures, and the decomposition method. Furthermore, sufficient conditions are suggested that under them, the problem has a unique optimal solution. Also, a comparison between the recently proposed algorithm and the known ones will be made.

1. Introduction

In this paper, the following mathematical model is considered:

$$\text{Minimize} \quad Z(x) = \sum_{j=1}^{n} c_j.x_j, \tag{1}$$

$$\text{Subject to} \quad x \in X\left(A, B, d^1, d^2\right) := \left\{x \in [0, 1]^n | Aox \geq d^1 \text{ and } Box \leq d^2\right\} \tag{2}$$

where $c_j \in R$ is the coefficient associated with the variable x_j; $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ and $l \times n$ fuzzy relation matrices with $0 \leq a_{ij}, b_{ij} \leq 1$, respectively; $d^1 = [d_1^1, \ldots, d_m^1]^{\mathrm{T}} \in [0, 1]^m$ and $d^2 = [d_1^2, \ldots, d_l^2]^{\mathrm{T}} \in [0, 1]^l$; and the operation "o" represents the max-Einstein composition operator.

Let $\underline{n}$ be the index set $\{1, \ldots, n\}$ for each positive integer number n. The constraint part of model (1) and (2) is to find a set of solution vectors $x \in [0, 1]^n$ such that

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$$\begin{aligned}
&\mathrm{Max}_{j\in \underline{n}}\left\{\frac{a_{ij}.x_j}{2-(a_{ij}+x_j-a_{ij}.x_j)}\right\}\geq d_i^1, \quad \forall i\in \underline{m},\\
&\mathrm{Max}_{j\in \underline{n}}\left\{\frac{b_{ij}.x_j}{2-(b_{ij}+x_j-b_{ij}.x_j)}\right\}\leq d_i^2, \quad \forall i\in \underline{l}.
\end{aligned} \tag{3}$$

Let $x^1 = [x_j^1]$ and $x^2 = [x_j^2]$ be two n-dimensional vectors. Define $x^1 \leq x^2$ if and only if $x_j^1 \leq x_j^2$ for all $j \in \underline{n}$, where $x^1, x^2 \in X(A, B, d^1, d^2)$. A solution $\bar{x} \in X(A, B, d^1, d^2)$ is called the maximum solution if $x \leq \bar{x}$ for all $x \in X(A, B, d^1, d^2)$. On the other hand, $\underline{x} \in X(A, B, d^1, d^2)$ is a minimal solution if $\forall x \in X(A, B, d^1, d^2)$, where $x \leq \underline{x}$ implies that $x = \underline{x}$. A solution $x^* \in X(A, B, d^1, d^2)$ is optimal for the problem (1) and (2) if $Z(x^*) \leq Z(x)$ for all $x \in X(A, B, d^1, d^2)$. In this paper, the notations $\bar{x}$ and $\underline{x}$ are specially applied to show the maximum and the minimal solutions of $X(A, B, d^1, d^2)$.

It is not difficult to see that fuzzy relation equations (FRE) can also be viewed as fuzzy relation inequalities (FRI). Consequently, FRE can be seen as a special case of FRI. FRE, FRI, and the problems related to them have been studied by many researchers since the resolution of FRE was proposed in [1] in 1976 (see for instance, Refs. [2–42,1,43–55,57]).

1.1. Applications of FRE and FRI

Applications of FRE, FRI, and the problems related to them can be seen in many areas, for instance, fuzzy decision-making, fuzzy symptom diagnosis, and especially fuzzy medical diagnosis [6,21,28,39,55]. The theory and applications of FRE developed up to 1989 were well documented in [8] in the first monograph on this issue. The most recent monograph on fuzzy relational equations and their applications is due to Peeva and Kyosev [42]. Good overviews can also be found in [9,12,13,24,40,41,55].

1.2. Relevant literature review and motivations

The problem of minimizing a linear objective function subject to a consistent system of max–min equations was first investigated by Fang and Li [10] in 1999. It was shown that this problem can be decomposed into two sub-problems by separating the negative and non-negative coefficients of the objective function, both of which are subject to the same constraints. The objective function with negative coefficients assumes its optimum at the maximum solution while the objective function with non-negative coefficients assumes its optimum at one of the minimal solutions which can be determined by a 0–1 integer programming problem. This 0–1 integer programming problem is solved by the branch-and-bound method with jump-tracking technique.

Wu et al. [51] considered Fang and Li's model. They enhanced Fang and Li's method by providing an efficient procedure that visits much fewer nodes in the solution tree than that of Fang and Li's procedure. Wu and Guu [52] proposed a necessary condition for an optimal solution to exist. Three rules for simplifying the work of computing an optimal solution are provided based on this necessary condition.

A more generalized case of Fang and Li's model in which the constraints are max–min fuzzy relation inequalities was considered in [56]. The optimization problem with one linear objective function and finitely many fuzzy relation inequality constraints is abbreviated as OLOFRIC. In the literature, an OLOFRIC problem is solved by converting it into two sub-problems according to the negative and non-negative coefficients in the objective function with the same constraints. These two sub-problems have their optimal values at the maximal point and one of the minimal points of the feasible domain, respectively. Since the feasible domain has a unique maximal point and finitely many minimal points, the known methods need to verify every minimal point of the feasible domain to obtain an optimal solution.

In order to determine the minimal points of the feasible domain, we should firstly find its quasi-minimal solutions using the set of all the FRI paths of the inequalities (3). Then its minimal solutions can be determined by pairwise comparison of those quasi-minimal solutions. Wang et al. [45] showed that when the components of vector d^1 satisfy the condition $d_1^1 > d_2^1 > \cdots > d_m^1$, the set of all the quasi-minimal solutions of FRI (3) with the max–min composition is equal to its set of all the minimal solutions. Zhang et al. [56] designed an algorithm to solve the problem (1) and (2) with the max–min composition under Wang et al.'s condition. Also, Guo and Xia [14] presented a necessary condition of optimality for this problem with the max–min composition. They proposed an algorithm to solve the problem based on the necessary condition and Wang et al.'s condition.

Unfortunately, Wang et al.'s condition for the max-Einstein composition is not true in a general case. For some large scale problems with this composition, too many quasi-minimal points have to be verified. This is a huge work. This suggests that generating all the minimal solutions of FRI is very difficult. Thus, designing an algorithm for determination of an optimal solution of the second sub-problem without explicitly generating all the quasi-minimal points is motivated.

Loetamonphong and Fang [25] studied Fang and Li's model with the max-product composition and they introduced some procedures to reduce the problem. They also decomposed the reduced problem into several sub-problems with smaller dimensions and solved them by the branch-and-bound method.

Loetamonphong and Fang's decomposition procedure is very interesting and useful to reduce the search domain of an optimal solution. We are interested the question of whether the procedure is true to the problem (1) and (2) or not. Thus the study of the topic is motivated.